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Novel Hamy Mean Aggregation Operators Based on Advanced Operations for T-Spherical Fuzzy Group Decision-Making

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ABSTRACT

Some advanced operation laws (AdOLs) are put forward with the T-spherical fuzzy (TSF) information. A series of novel Hamy mean aggregation operators are developed, specifically TSF advanced Hamy mean (TSFAdHM), TSF advanced dual Hamy mean (TSFAdDHM), TSF advanced weighted Hamy mean (TSFAdWHM) and TSF advanced weighted dual Hamy mean (TSFAdWDHM) operators. These aggregation operators integrate the strengths of Hamy mean and AdOLs in that they not only reveal correlations among multi-input variables but also eliminate counter-intuitive problems, as well as reflect decision-maker risk preferences. Several reasonable natures and peculiar types of these aggregation operators are investigated. Moreover, a novel group decision-making framework for the type of TSF aggregation operator is constructed. The tractability and usefulness of the decision-making model are examined by some numerical examples. The sensitivity and method comparison analyses are used to demonstrate that the proposed model can remedy the drawbacks of existing methods, and this article offers a very useful method for the complicated group decision-making issues.

1. Introduction

The MAGDM (multi-attribute group decision-making) is an influential stream of contemporary decision-making science, the essence of which is a process of rationally ranking a finite collection of options regarding each attribute and selecting an optimal option, which can address significantly decision-making problems in the area of economics, society, engineering, energy, and so on[1-3]. The information aggregation type decision-making methodology is a significant part of the response to the MAGDM challenges, and its key problem lies in how to accurately convey people's subjective judgmental information and how to efficiently aggregate the attribute evaluation information. For

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this reason, group decision-making problems have been extensively studied by numerous investigators [4-11].

In general, the evaluation values given by decision-makers always contain vague and uncertain information [12].To characterize them more accurately and comprehensively, a series of extended fuzzy sets with binary and ternary structures have been evolved on the basis of traditional fuzzy sets, respectively. The q-rung orthopair fuzzy set is a nice means with a generalized binary structure capable of efficiently representing evaluation information, characterized by a q-power sum of membership and non-membership degrees (MD, ND) not exceeding one [13].There are several specific special cases when q takes different values, i.e., intuitionistic fuzzy set (q=1) [14], Pythagorean fuzzy set (q=2)[15] and Fermatean fuzzy set (q=3)[16].

Unfortunately, subjective information regarding abstinence and refusal is often objectively present when people judge objects in the reality decision-making process, such as in the voting scenario [17], then relying only on grades of affiliation and non-affiliation to describe the evaluation information is not enough. For this reason, Cuong [17] introduced the notion of picture fuzzy set containing grades of affiliation, abstinence (AD) and non-affiliation, but it still suffers from the constraint that the total of three grades does not exceed one. Subsequently, Mahmood et al. [18] first presented the spherical fuzzy set and TSFS respectively as two concepts. The TSFS is a generalized ternary structure that fulfills the three affiliation functions whose sum of q-th power does not exceed one. Compared to existing extended fuzzy sets, TSFS, as the most generalized expression tool, has the greatest advantage that it has a more powerful decision space, breaking the constraints of the three grades of membership distributions and enabling freer expression of personal subjective judgments and preferences [19-20].

Currently, scholars have designed various aggregation operators for TSFS. The operational rules of TSF numbers (TSFNs) are an extremely important foundation for the aggregation operators' development. Here are the operational laws of numerous existing TSFNs, such as Algebraic [21], Hamacher [22], Einstein [23], interactive [24-25], Frank [26], Dombi [27], Schweizer-Sklar (SS) [28], Aczel-Alsina (AA) [29-30], etc. Among them, the Algebraic, Hamacher and Einstein are all operational rules that do not contain decision adjustable parameters. There is also no parameter in interactive operational laws, but this law emphasizes the interaction between arbitrary two TSFNs to prevent counterintuitive results when zero value occurs in any of the three grades of membership. The Frank, Dombi, SS, AA, etc. are t-norms operational types, which can increase the versatility of the information fusion operation. In addition, the above operational laws can be integrated with Bonferroni mean (BM), Heronian mean (HeM), Hamy mean (HM), Maclaurin symmetric mean (MSM), Muirhead mean (MM), etc. to develop a variety of novel aggregation operators in the TSF environment, respectively. For example, Wang and Zhang [25] designed the new interaction power HeM (TSFIPHeM) aggregation operator with power integrated with the HeM and considering the interaction of TSFNs. Naseem et al. [31] integrated power with MSM to design TSF power MSM (TSFPMSM) aggregation operators. According to the strengths of power and MM, Liu et al. [19] raised the power MM (TSFPMM) operator and its dual shape for TSFS. Yang and Pang [32] advanced the TSF Dombi BM (TSFDBM) aggregation operators by extending Dombi t-norms and BM in the TSF environment. Based on AAOLs, Wang et al. [29] developed TSF AA weighted HM (TSFAAWHM) operator and its dual form.

However, the following two issues need to be further addressed in the MAGDM solution process, as modern decision-making problems become more complex. (1) Various types of TSF aggregation operators are designed based on the above numerous operational rules, but these laws still have some drawbacks. For example, suppose that an assortment of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$,where *i*=1,2,⋯,*n*, If there are arbitrary TSFN $\delta_k=(\tau_k,\eta_k,0)$, $\tau_k\neq0,\eta_k\neq0$, and then using the operation rules related to TSFNs respectively. We find that: using the Algebraic [21], Hamacher [22], Einstein [23] and SS [28], the ND in the TSFNs obtained by aggregation is still zero-valued, which means that the TSFN with $v_k=0$ in the process of computation erases the influence of ND in other TSFNs on the set result ND, and even distorts the decision-makers' judgment intention. Therefore, these arithmetic rules are irrational and can lead to counterintuitive results. Aggregation of information was carried out using the operational rules of Frank [26], Dombi [27], AA [29-30], but the results could not be obtained. This is because it is not possible to avoid a situation where the truth of the logarithm has a zero value and the denominator has a zero value, resulting in the inability to obtain meaningful aggregation results. Unlike the previous rules, the IOLs [24-25] utilize the interaction between MD, AD and ND so that the aggregated ND is not zero-valued, thus realizing the elimination of counterintuitive phenomena during the aggregation operation. Despite this advantage, the IOLs are computationally complex compared to algebraic rules, especially when combined with Frank, HeM, and BM and so on, the aggregation operation is more complicated. Therefore, a novel operational rule for TSFNs is needed here. (2) In many real-world decision-making problems, there are inherent correlations between some attributes that are often considered by scholars. In TSF environment, the HeM and BM-based aggregation operators can only capture the interrelationship between any two attributes, while the HM, MM and MSM can capture the interrelationship of multiple attributes. In comparison, HM is considered as an extended mathematical form of MSM and is relatively simple to express mathematically [33-34], but the existing TSFAAHM aggregation operator cannot deal with the decision problem of TSFNs whose elements contain zero values. Therefore, we should focus on the HM operator that can capture multiple attribute associations with a novel function structure to endow the HM operator with more powerful information fusion capability.

In the IFS and q-ROFS settings, the AdOLs proposed by Kumar and Chen [35-37], respectively, can be used to deal with decision problems with zero-valued MDs or NDs, but the parameter ($0<\varepsilon<1$) in these laws has not been discussed, as well as these laws have not been studied in integration with correlation-capable operators. Hence, we are inspired by Kumar and Chan's series of works [35-37], firstly we extend and discuss AdOLs in the context of TSFS. Secondly, since the HM is more capable of capturing interrelationships than the HeM and BM, and the HM is superior to MSM and MM in terms of mathematical results and computational complexity. Therefore, we focus on the HM and develop novel TSF set operators based on AdOLs. Since the aforementioned, the aim of this article is to design TSF set operators integrated with AdOLs via HM and then use them to settle MAGDM issues. Specifically, the series TSFAdWHM and its dual shape are put forward, and a novel TSF MAGDM model according to the designed aggregation operators is built to resolve complex MAGDM issues.

The primary proffers of this article are presented as below: (1) some AdOLs for TSFNs are defined. These rules contain a parameter ($0<\varepsilon<1$) that can handle the case of zero-valued elements in TSFNs, which can effectively eliminate counterintuitive phenomena. Compared with the IOLs, the proposed rules are not only less computationally complex but also responsive to the decision-makers' preference. (2) Novel TSF aggregation operators are developed. Specifically, there are the TSFAdHM operator and its dual form, as well as their weighted forms. These operators not only enable the elimination of counterintuitive phenomena but also consider correlations among multiple attributes. These operators are also shown to be more generalized and flexible than the current aggregation operators with TSF information, as well as reflecting the decision-makers' decision preferences. (3) An aggregation operator-based MAGDM model is built in TSF setting, which has the advantages of being stronger, more flexible and less computationally complex than the existing aggregation operator based MAGDM methodologies.

Section 2 briefly reviews the basic notions of TSF and HM. The AdOLs of TSFNs are proposed, and a series of TSFAdHM aggregation operators are exploited in Section 3. Section 4 presents an innovative TSF MAGDM model relying on the developed aggregation operators. Section 5 tests some numerical examples to explicate the feasibility, practicality and advantages of the proposed method. Finally, Section 6 presents the conclusion.

2. Preliminaries

Some basic notions of TSFS, HM and DHM are concisely introduced under this section.

2.1 TSFS

Definition 1[18] Suppose a given argument domain is Y, and call

$$\Delta = \{< y, \Delta(\tau_\Delta(y), \eta_\Delta(y), \nu_\Delta(y)) > \ | \, y \in Y\} \tag{1}$$

an TSFS on Y, where $\tau_\Delta(y)$, $\eta_\Delta(y)$ and $\nu_\Delta(y)$ denote the grades of affiliation, abstinence and non-affiliation of y belonging to Δ, respectively. They satisfy $0 \le \tau_\Delta(y)$, $\eta_\Delta(y)$, $\nu_\Delta(y) \le 1$, and $0 \le (\tau_\Delta(y))^q + (\eta_\Delta(y))^q + (\nu_\Delta(y))^q \le 1$, $q \ge 1$, $y \in Y$. Moreover, the hesitancy level of y belonging to Δ is denoted as $\pi_\Delta(y) = \sqrt[q]{1 - (\tau_\Delta(y))^q - (\eta_\Delta(y))^q - (\nu_\Delta(y))^q}$. For ease of calculation, $\delta = (\tau_\delta, \eta_\delta, \nu_\delta)$ is referred to as a TSFN.

Definition 2 [24] Suppose $\delta = (\tau_\delta, \eta_\delta, \nu_\delta)$ is any TSFN, then the score value $s(\delta)$ and the accuracy value $a(\delta)$ for δ are expressed below.

$$s(\delta) = \frac{1 + (\tau_\delta)^q - (\eta_\delta)^q - (\nu_\delta)^q}{2} \tag{2}$$

$$a(\delta) = (\tau_\delta)^q + (\eta_\delta)^q + (\nu_\delta)^q \tag{3}$$

where $s(\delta) \in [0,1]$, $a(\delta) \in [0,1]$.

Based on the above formulas, an approach for TSFN size comparison is presented. Suppose arbitrary two TSFNs are δ_1 and δ_2, then: (1) if $s(\delta_1)$ is greater than $s(\delta_2)$, then δ_1 is superior to δ_2; (2) if $s(\delta_1)$ and $s(\delta_2)$ are equal, then:

i) if $a(\delta_1)$ is more than $a(\delta_2)$, then δ_1 is better than δ_2;

ii) if $a(\delta_1)$ equal $a(\delta_2)$, then δ_1 is similar to δ_2.

Definition 3 [18] Let $\delta = (\tau, \eta, \nu)$, $\delta_1 = (\tau_1, \eta_1, \nu_1)$ and $\delta_2 = (\tau_2, \eta_2, \nu_2)$ be the three TSFNs. The basic rules of operation are outline below ($\lambda > 0$, $q \ge 1$):

(1) $\delta_1 \oplus \delta_2 = \left(\sqrt[q]{1 - \prod_{i=1}^{2} (1 - \tau_i^q)}, \prod_{i=1}^{2} \eta_i, \prod_{i=1}^{2} \nu_i \right)$;

(2) $\delta_1 \otimes \delta_2 = \left(\prod_{i=1}^{2} \tau_i, \sqrt[q]{1 - \prod_{i=1}^{2} (1 - \eta_i^q)}, \sqrt[q]{1 - \prod_{i=1}^{2} (1 - \nu_i^q)} \right)$;

(3) $\lambda\delta = \left(\sqrt[q]{1-(1-\tau^q)^\lambda}, \eta^\lambda, \nu^\lambda\right)$;

(4) $\delta^\lambda = \left(\tau^\lambda, \sqrt[q]{1-(1-\eta^q)^\lambda}, \sqrt[q]{1-(1-\nu^q)^\lambda}\right)$.

Example 1 Let δ_1=(0.5,0.7,0.5) and δ_2=(0.4,0.6,0.8) be two TSFNs (q=3). We can get (0.200, 0.786, 0.831) by using $\delta_1 \otimes \delta_2$ calculation, but $(0.200)^3+(0.786)^3+(0.831)^3=1.066>1$, which does not satisfy the constraint of TSFN. Consequently, it is unreasonable. Assuming also that δ_3=(0.5,0.7,0.0) and δ_4=(0.4,0.0,0.8) are two TSFNs (q=3), $\delta_1 \oplus \delta_2$ is employed and we can obtain the result as (0.566, 0.000, 0.000). Obviously, 0.7 in δ_3 and 0.8 in δ_4 do not play a role in the result. This counterintuitive result can show that the basic operational laws of TSFNs are irrational.

From the results of Example 1, we can find that Definition 3 is unreasonable. For the problem of counterintuitive results, the TSF interactive operational laws (IOLs) raised by Ju et al. [24] on the basis of the interactional operation of IFNs by He et al.[38] can effectively eliminate it. The IOLs of TSFNs are presented as follows.

Definition 4 [24] Let $\delta=(\tau, \eta, \nu)$, $\delta_1=(\tau_1, \eta_1, \nu_1)$ and $\delta_2=(\tau_2, \eta_2, \nu_2)$ be three TSFNs, then their IOLs are represented as below ($\lambda>0$, $q\geq1$).

(1) $\delta_1 \oplus \delta_2 = \left(\sqrt[q]{1-\prod_{i=1}^2(1-\tau_i^q)}, \sqrt[q]{\prod_{i=1}^2(1-\tau_i^q)-\prod_{i=1}^2(1-\tau_i^q-\eta_i^q)}, \sqrt[q]{\prod_{i=1}^2(1-\tau_i^q-\eta_i^q)-\prod_{i=1}^2(1-\tau_i^q-\eta_i^q-\nu_i^q)}\right)$;

(2) $\delta_1 \otimes \delta_2 = \left(\sqrt[q]{\prod_{i=1}^2(1-\nu_i^q-\eta_i^q)-\prod_{i=1}^2(1-\nu_i^q-\eta_i^q-\tau_i^q)}, \sqrt[q]{\prod_{i=1}^2(1-\nu_i^q)-\prod_{i=1}^2(1-\nu_i^q-\eta_i^q)}, \sqrt[q]{1-\prod_{i=1}^2(1-\nu_i^q)}\right)$;

(3) $\lambda\delta = \left(\sqrt[q]{1-(1-\tau^q)^\lambda}, \sqrt[q]{(1-\tau^q)^\lambda-(1-\tau^q-\eta^q)^\lambda}, \sqrt[q]{(1-\tau^q-\eta^q)^\lambda-(1-\tau^q-\eta^q-\nu^q)^\lambda}\right)$;

(4) $\delta^\lambda = \left(\sqrt[q]{(1-\eta^q-\nu^q)^\lambda-(1-\tau^q-\eta^q-\nu^q)^\lambda}, \sqrt[q]{(1-\nu^q)^\lambda-(1-\eta^q-\nu^q)^\lambda}, \sqrt[q]{1-(1-\nu^q)^\lambda}\right)$.

For the IOLs for TSFNs, Yang and Pang [39] (2022) argued that these laws are degenerated into the following AOLs for TSFNs if the interactions among the three elements of TSFNs are not considered.

(1) $\delta_1 \oplus \delta_2 = \left(\sqrt[q]{1-\prod_{i=1}^2(1-\tau_i^q)}, \prod_{i=1}^2\eta_i, \prod_{i=1}^2\nu_i\right)$;

(2) $\delta_1 \otimes \delta_2 = \left(\prod_{i=1}^2\tau_i, \prod_{i=1}^2\eta_i, \sqrt[q]{1-\prod_{i=1}^2(1-\nu_i^q)}\right)$;

(3) $\lambda\delta = \left(\sqrt[q]{1-(1-\tau^q)^\lambda}, \eta^\lambda, \nu^\lambda\right)$;

(4) $\delta^{\lambda}=\left(\tau^{\lambda},\eta^{\lambda},\sqrt[q]{1-(1-\nu^{q})^{\lambda}}\right)$.

We use $\delta_1\oplus\delta_2$ and $\delta_1\otimes\delta_2$ in the AOLs of TSFNs to compute δ_1=(0.5,0.7,0.5) and δ_2=(0.4,0.6,0.8) in Example 1, we get (0.566, 0.420,0.400) and (0.200,0.420,0.831) respectively, then $(0.566)^3+(0.420)^3+(0.400)^3=0.319<1$ and $(0.200)^3+(0.420)^3+(0.831)^3=0.655<1$, which the results satisfy the constraints of TSFN, so these TSFNs' AOLs are reasonable. However, the AOLs still fail to eliminate the counterintuitive phenomenon. For this reason, this paper needs to develop new AdOLs for TSFNs based on the works of Kumar and Chen [35-37].

2.2 Hamy mean and dual Hamy mean

Hara et al.[40] first proposed the HM for crisp numbers, which can be used to capture the correlationship among multiple arguments.

Definition 5[40] Suppose a_i (i=1, 2,…, n) is a assortment of crisp values, and k=1, 2,…,n, if

$$HM^{(k)}(a_1,a_2,\ldots,a_n)=\frac{\sum_{1\le i_1<\cdots<i_k\le n}\left(\prod_{j=1}^{k}a_{i_j}\right)^{1/k}}{C_n^k} \tag{4}$$

Then HM$^{(k)}$ is referred the Hamy mean, where C_n^k is a combinatorial number and ($i_1,i_2,\ldots,i_k$) is a k-metric permutations of (1,2,…,n).

Obviously, the HM is a Schurconvex and monotonic when fusing numerical information. The dual HM such that it also satisfies Schurconvexity and monotonicity is presented by Wu et al.[41].

Definition 6[41] Suppose a family of crisp numbers is a_i (i=1, 2,…, n), and k=1, 2,…,n, if

$$DHM^{(k)}(a_1,a_2,\ldots,a_n)=\left(\prod_{1\le i_1<\cdots<i_k\le n}\frac{\sum_{j=1}^{k}a_{i_j}}{k}\right)^{1/C_n^k} \tag{5}$$

Then DHM$^{(k)}$ is referred the dual Hamy mean, where C_n^k is a combinatorial number and (i_1, i_2,…, i_k) is a k-metric permutations of (1,2,…,n).

3 TSFAdHM Aggregation Operators

Some AdOLs are introduced in TSF environment. According to these laws, we herein develop a family of the TSFAdHM aggregation operators, i.e., the TSFAdHM, TSFAdWHM, TSFAdDHM, and TSFAdWDHM.

3.1 AdOLs of TSFNs

The existing AdOLs in the IFS and q-ROFS environments can only handle binary-structured data, but they are not applicable to the TSFNs. For this reason, we define new AdOLs for TSFNs in the TSFS context as follows:

Definition 7 Let $\delta=(\tau,\eta,\nu)$, $\delta_1=(\tau_1,\eta_1,\nu_1)$ and $\delta_2=(\tau_2,\eta_2,\nu_2)$ be three TSFNs. Then the AdOLs of TSFNs are described as below (λ>0, q≥1, 0<ε<1).

(1) $\delta_1 \oplus \delta_2 = \left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon\tau_i^q)\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon(1-\eta_i^q))\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon(1-\nu_i^q))\right)} \right)$;

(2) $\delta_1 \otimes \delta_2 = \left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon(1-\tau_i^q))\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon(1-\eta_i^q))\right)}, \sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{2}(1-\varepsilon\nu_i^q)\right)} \right)$;

(3) $\lambda\delta = \left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-(1-\varepsilon\tau^q)^{\lambda}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-(1-\varepsilon(1-\eta^q))^{\lambda}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-(1-\varepsilon(1-\nu^q))^{\lambda}\right)} \right)$;

(4) $\delta^{\lambda} = \left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-(1-\varepsilon(1-\tau^q))^{\lambda}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-(1-\varepsilon(1-\eta^q))^{\lambda}\right)}, \sqrt[q]{\frac{1}{\varepsilon}\left(1-(1-\varepsilon\nu^q)^{\lambda}\right)} \right)$.

Theorem 1 Let $\delta_1=(\tau_1, \eta_1, \nu_1)$ and $\delta_2=(\tau_2, \eta_2, \nu_2)$ be two TSFNs, and they satisfy the following operation properties (λ, λ_1, $\lambda_2>0$):

(1) $\delta_1 \oplus \delta_2 = \delta_2 \oplus \delta_1$;
(2) $\delta_1 \otimes \delta_2 = \delta_2 \otimes \delta_1$;
(3) $\lambda(\delta_1 \oplus \delta_2) = \lambda\delta_1 \oplus \lambda\delta_2$;
(4) $\lambda_1\delta_1 \oplus \lambda_2\delta_1 = (\lambda_1+\lambda_2)\delta_1$;
(5) $(\delta_1 \otimes \delta_2)^{\lambda} = (\delta_1)^{\lambda} \otimes (\delta_2)^{\lambda}$;
(6) $\delta_1^{\lambda_1} \otimes \delta_1^{\lambda_2} = \delta_1^{(\lambda_1+\lambda_2)}$.

The above operation properties are readily proved. The proof is omitted.

Remark 1 Let $\delta_i=(\tau_i, \eta_i, \nu_i)$ be a collection of TSFNs, then their arithmetic mean (AM) and geometric mean (GM) operations based on the Definition 7 can be expressed as follows:

$$\frac{1}{n}\oplus_{i=1}^{n}\delta_i = \left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)} \right) \tag{6}$$

$$\frac{1}{n}\otimes_{i=1}^{n}\delta_i = \left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\tau_i^q))^{1/n}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}\right)}, \sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\nu_i^q)^{1/n}\right)} \right) \tag{7}$$

Theorem 2 Let $\delta_i=(\tau_i, \eta_i, \nu_i)$ be an assortment of TSFNs, where i=1,2,…,n, and 0<ε<1. Then the score function of the TSF AM (Eq.(6)) based on AdOLs increases monotonically with ε, while the score function of the TSF GM (Eq.(7)) decreases monotonically with ε.

Proof: From Eq.(6), we have that

$$AM = \left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}\right)}, \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)} \right)$$

In order to prove that the score function of TSF AM is monotonically increasing regarding ε, we only need to show that s(AM) is increasing in monotonically on ε. Then, based on Eq. (2), we can get

$$s(AM) = \frac{1}{2}\left(\frac{1}{\varepsilon}\left(3-\left(\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)\right)-1\right)$$

Let $f(\varepsilon)=\frac{1}{\varepsilon}\left(3-\left(\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)\right)$, we carry out the derivation regarding ε for f(ε), from which we get

$$\frac{df(\varepsilon)}{d\varepsilon}=\frac{d}{d\varepsilon}\left(\frac{1}{\varepsilon}\left(3-\left(\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)\right)\right)$$
$$=\frac{1}{\varepsilon}\left(\begin{array}{l}\frac{1}{n}\left(\prod_{i=1}^{n}\tau_i^q(1-\varepsilon\tau_i^q)^{(1/n)-1}+\prod_{i=1}^{n}(1-\eta_i^q)(1-\varepsilon(1-\eta_i^q))^{(1/n)-1}+\prod_{i=1}^{n}(1-\nu_i^q)(1-\varepsilon(1-\nu_i^q))^{(1/n)-1}\right)\\-\frac{1}{\varepsilon}\left(3-\left(\prod_{i=1}^{n}(1-\varepsilon\tau_i^q)^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\eta_i^q))^{1/n}+\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i^q))^{1/n}\right)\right)\end{array}\right)$$

Since s(AM)$\in$[0,1], then 0≤f(ε)≤3. And since (1/n)-1<0, $\tau_i^q(1-\varepsilon\tau_i^q)^{(1/n)-1}>1$, $(1-\eta_i^q)(1-\varepsilon(1-\eta_i^q))^{(1/n)-1}>1$, $(1-\nu_i^q)(1-\varepsilon(1-\nu_i^q))^{(1/n)-1}>1$, then

$$\frac{1}{n}\left(\prod_{i=1}^{n}\tau_i^q(1-\varepsilon\tau_i^q)^{(1/n)-1}+\prod_{i=1}^{n}(1-\eta_i^q)(1-\varepsilon(1-\eta_i^q))^{(1/n)-1}+\prod_{i=1}^{n}(1-\nu_i^q)(1-\varepsilon(1-\nu_i^q))^{(1/n)-1}\right)>3$$

We get $\frac{df(\varepsilon)}{d\varepsilon}>0$, then f(ε) is monotonically increasing about ε. Thus, the score function of TSF AM based on AdOLs is monotonically increasing about ε.

For the TSF GM score function, it is easy to show that it is decreasing monotonically on ε.

Remark 2 The score functions of TSF AM and GM based on AdOLs have monotonicity when 0<ε<1, which means that the judges have the flexibility to adopt ε value that is realistic for decision-making according to his/her attitude. For the TSF AM, the more pessimistic the decision-maker is, the closer to zero the value of ε is; conversely, the closer to one the value of ε is.

Example 2 Let δ_1=(0.8,0.5,0.0), δ_2=(0.7,0.3,0.7) and δ_3=(0.4,0.0,0.9) be three TSFNs. We calculate the TSF AM and GM using the AOLs [39], the IOLs [24] and the AdOLs of TSFNs, respectively. The results, as well as a comparison of the characteristics of the three laws, are listed in Table 1.

We can find from the results of various operational laws in Table 1 that the AM and GM calculations based on AOLs [39] appear to be counterintuitive with AD, ND getting zero value in TSFN. The score function value of AOL-AM in Fig.1 is significantly higher than the other score values, while the remaining two algorithms are not counterintuitive. Thus, AOLs [39] cannot eliminate counterintuitive phenomena, but IOLs [24] and AdOLs have this ability. In terms of computational complexity, the computation of AOLs is both straightforward and simple, with MD, AD and ND in TSFNs operating independently of each other, whereas IOLs [24] are more complex, with interactions between TSFNs taken into account in both AM and GM computations. The computational complexity of AdOLs is in between. In terms of operation flexibility, the AOLs and IOLs do not contain any arithmetic parameter, while AdOLs have a parameter ε, which can be calculated by taking the value in the range of (0, 1), and in Example 2 with ε=0.99. From Fig. 1, the value of the AdOL-AM (AdOL-GM) score function increases (decreases) as ε becoming larger. The judges can adopt the ε value depending on the actual decision-making scenario. The monotonicity of AdOL-AM and AdOL-GM score function values also indicate the decision-makers' attitude and preference.

Table 1 Comparison of different operational laws (q=3)

Results and features		AOLs[39]	IOLs[24]	AdOLs (ε=0.99)
Results	AM	(0.6914,0.0000,0.0000)	(0.6914,0.4147,0.6839)	(0.6911,0.2996,0.5040)
	GM	(0.6073,0.0000,0.7591)	(0.6319,0.3174,0.7591)	(0.6118,0.2996,0.7582)
Eliminating counter-intuition		No	Yes	Yes
Computational complexity		Simple	Complex	Medium
Operational flexibility		No	No	Yes
Decision-making preferences		No	No	Yes

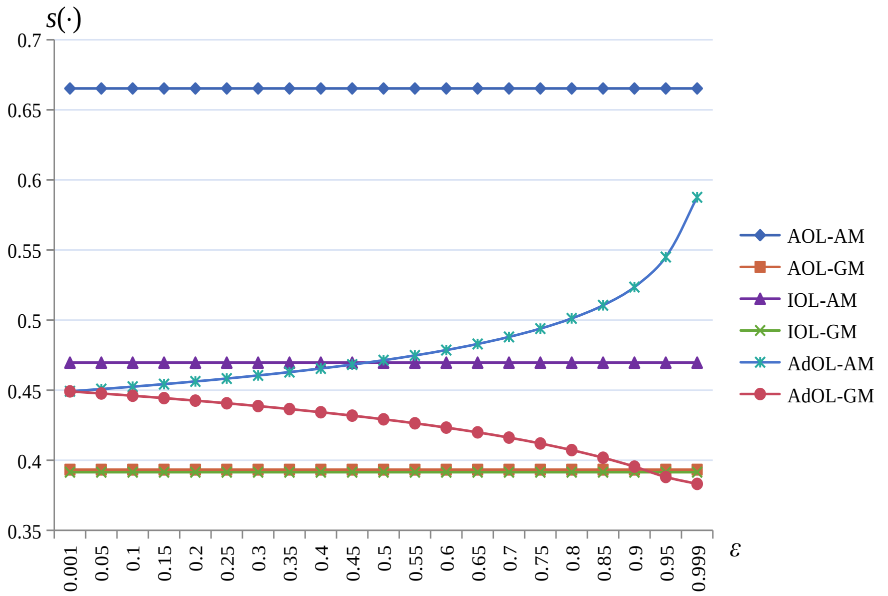


Fig.1 Variation of each AM and GM score value with ε

3.2 TSFAdHM Operator

In this subsection, we integrate HM with the AdOLs of TSFNs to develop new TSFAdHM operator, and some desirable properties are discussed.

Definition 8 Let a set of TSFNs be $\delta_i=(\tau_i,\eta_i,v_i)$ (i=1,2,⋯,n), then the TSFAdHM operator is defined as (k=1,2,⋯,n),

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\dots,\delta_n)=\frac{\underset{1\le i_1<\cdots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}a_{i_j}\right)^{1/k}}{C_n^k} \tag{8}$$

Where C_n^k is a combinatorial number and (i_1, i_2,…, i_k) is a k-metric permutations of (1,2,…,n).

Since the AdOLs for TSFNs in Section 3.1, we can obtain the following theorem.

Theorem 3 Suppose a set of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$, where i=1,2,…,n, then the outcome derived from Definition 8 is expressed as (k=1,2,…,n)

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))^{1/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)}\end{array}\right) \tag{9}$$

Proof: Based on the AdOLs of TSFNs, we can get

$$\otimes_{j=1}^{k}\delta_{i_j}=\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})\right)}\right)$$

Then,

$$\left(\otimes_{j=1}^{k}\delta_{i_j}\right)^{1/k}=\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))^{1/k}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})^{1/k}\right)}\right)$$

And

$$\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\delta_{i_j}\right)^{1/k}=\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))^{1/k}\right)\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})^{1/k}\right)\right)}\end{array}\right)$$

Subsequently, we have

$$\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\delta_{i_j}\right)^{1/k}}{C_n^k}=\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))^{1/k}\right)\right)^{1/C_n^k}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)}\end{array}\right)$$

Therefore, Eq. (9) holds.

Let an assortment of TSFNs be $\delta_i=(\tau_i, \eta_i, \nu_i)$, where i=1,2,⋯,n, the TSFAdHM operator is endowed with these desirable natures below.

Theorem 4 (Idempotency) If all TSFNs are the same, i.e., $\delta_i=\delta$ for all i, then

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\delta \tag{10}$$

Proof: Since δ_i=δ for all i, we can obtain

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\delta_{i_j}\right)^{1/k}}{C_n^k}=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\delta\right)^{1/k}}{C_n^k}=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\delta^{k}\right)^{1/k}}{C_n^k}=\frac{C_n^k\delta}{C_n^k}=\delta$$

Theorem 5 (Commutativity) If ($\sigma_1,\sigma_2,\ldots,\sigma_n$) is any substitution of ($\delta_1,\delta_2,\ldots,\delta_n$), then

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=TSFAdHM^{(k)}(\sigma_1,\sigma_2,\ldots,\sigma_n) \tag{11}$$

Proof: Since ($\sigma_1,\sigma_2,\ldots,\sigma_n$) is any substitution of ($\delta_1,\delta_2,\ldots,\delta_n$), we can get

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\delta_{i_j}\right)^{1/k}}{C_n^k}=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\otimes_{j=1}^{k}\sigma_{i_j}\right)^{1/k}}{C_n^k}=TSFAdHM^{(k)}(\sigma_1,\sigma_2,\ldots,\sigma_n)$$

The TSFAdHM operator is degenerated into different special cases when the parameter k is assigned different values. Even in different decision-making environments, various special cases are obtained.

Case 1 If k=1, the TSFAdHM operator is reduced to a TSFAM operator.

$$\begin{aligned}&TSFAdHM^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=\\&\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon(1-\tau_{i_j}^q))^{1/1}\right)\right)^{1/C_n^1}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon\eta_{i_j}^q)^{1/1}\right)\right)^{1/C_n^1}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon\nu_{i_j}^q)^{1/1}\right)\right)^{1/C_n^1}\right)}\end{array}\right)\\&=\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon\tau_{i_1}^q\right)^{1/n}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon(1-\eta_{i_1}^q)\right)^{1/n}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon(1-\nu_{i_1}^q)\right)^{1/n}\right)}\end{array}\right)=\frac{\underset{1\le i_1\le n}{\oplus}\delta_{i_1}}{n}=\frac{\oplus_{i=1}^{n}\delta_i}{n}=TSFAM(\delta_1,\delta_2,\ldots,\delta_n)\end{aligned}$$

This operator is the TSF AM (Eq.(6)) in Remark 1.

Case 2 If k=n, the TSFAdHM operator is degenerated to a TSF advanced geometric (TSFAdG) operator.

$$\begin{aligned}&TSFAdHM^{(n)}(\delta_1,\delta_2,\ldots,\delta_n)=\\&\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{n}(1-\varepsilon(1-\tau_{i_j}^q))^{1/n}\right)\right)^{1/C_n^n}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{n}(1-\varepsilon\eta_{i_j}^q)^{1/n}\right)\right)^{1/C_n^n}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{n}(1-\varepsilon\nu_{i_j}^q)^{1/n}\right)\right)^{1/C_n^n}\right)}\end{array}\right)\\&=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\tau_i^q))^{1/n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\eta_i^q)^{1/n}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\nu_i^q)^{1/n}\right)}\end{array}\right)=\left(\otimes_{i=1}^{n}\delta_i\right)^{1/n}=TSFAdG(\delta_1,\delta_2,\ldots,\delta_n)\end{aligned}$$

Case 3 If k =1, the TSFAdHM operator is degenerated to a q-rung orthopair fuzzy advanced averaging (q-ROFAdA) operator assuming that the ADs of $\delta_i$are all zero (i.e., η=0).

$$TSFAdHM_{\eta=0}^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=$$
$$\left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon(1-\tau_{i_j}^q))^{1/1}\right)\right)^{1/C_n^1}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon v_{i_j}^q)^{1/1}\right)\right)^{1/C_n^1}\right)}\right)$$

$$=\left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon\tau_{i_1}^q\right)\right)^{1/n}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon(1-v_{i_1}^q)\right)\right)^{1/n}\right)}\right)=\frac{\oplus_{i=1}^{n}\delta_i}{n}=q-ROFAdA(\delta_1,\delta_2,\ldots,\delta_n)$$

Obviously, the q-ROFAdA is a particular form of the q-ROFWA developed by Kumar and Chen [36] when δ_i has weight 1/n, where i=1, 2,⋯,n.

Case 4 If q=1and k=1, the TSFAdHM operator is degenerated to a intuitionistic fuzzy advanced averaging (IFAdA) operator assuming that the ADs of δ_i are all zero (i.e., η=0).

$$TSFAdHM_{q=1,\eta_i=0}^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=$$
$$\left(\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon(1-\tau_{i_j}))^{1/1}\right)\right)^{1/C_n^1}\right),1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon v_{i_j})^{1/1}\right)\right)^{1/C_n^1}\right)\right)$$
$$=\left(\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon\tau_{i_1}\right)\right)^{1/n}\right),1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon(1-v_{i_1})\right)\right)^{1/n}\right)\right)=\frac{\oplus_{i=1}^{n}\delta_i}{n}=IFAdA(\delta_1,\delta_2,\ldots,\delta_n)$$

Clearly, the IFAdA operator is a special form of the advanced interval-valued intuitionistic fuzzy averaging (AIVIFA) operator proposed by Kumar and Chen [35].

3.3 TSFAdWHM Operator

Each input argument has different level of importance in many practical scenarios, and thus each input argument needs to be assigned a reasonable weight. To this end, we introduce the importance of each aggregation argument based on the above TSFAdHM operator, which can be obtained in its weighted form (TSFAdWHM).

Definition 9 Suppose a set of TSFNs is δ_i=(τ_i, η_i, v_i) (i=1,2,…,n), the corresponding weight of δ_i is w_i, meeting 0≤w_i≤1, and $\sum_{i=1}^{n}w_i=1$, then the TSFAdWHM operator is defined as (k=1,2,…,n)

$$TSFAdWHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\frac{\oplus_{1\le i_1<\cdots<i_k\le n}\left(\otimes_{j=1}^{k}(a_{i_j})^{w_{i_j}}\right)^{1/k}}{C_n^k} \tag{12}$$

Where C_n^k is a combinatorial number and (i_1, i_2,…, i_k) is a k-metric permutations of (1, 2,…,n).

Theorem 6 Let δ_i=(τ_i, η_i, v_i) (i=1, 2,…,n) be a family of TSFNs, the corresponding weight of δ_i be w_i, meeting 0≤w_i≤1, and $\sum_{i=1}^{n}w_i=1$, then the TSFAdWHM operator expansion form is represented as (k=1, 2,…,n)

$$TSFAdWHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\begin{pmatrix}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\tau_{i_j}^{q}))^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\nu_{i_j}^{q})^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)}\end{pmatrix} \tag{13}$$

Proof: According to the AdOLs of TSFNs, we can have

$$\left(\delta_{i_j}\right)^{w_{i_j}}=\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-(1-\varepsilon(1-\tau_{i_j}^{q}))^{w_{i_j}}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-(1-\varepsilon\eta_{i_j}^{q})^{w_{i_j}}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-(1-\varepsilon\nu_{i_j}^{q})^{w_{i_j}}\right)}\right)$$

The next proof procedure is similar to the proof of Eq. (9), and subsequently we can obtain that Eq. (13) holds.

Example 3 Suppose δ_1=(0.8,0.3,0.4), δ_2=(0.6,0.7,0.2) and δ_3=(0.9,0.1,0.5) are three TSFNs, their appropriate weights are 0.27,0.39 and 0.34. Then we employ the TSFAdWHM operator (assuming k=2) to fuse them and obtain an aggregated value. Its detailed calculation procedure is as follows (q=3, ε=0.99).

$$TSFAdWHM^{(2)}(\delta_1,\delta_2,\delta_3)=\begin{pmatrix}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon(1-\tau_{i_j}^{q}))^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon\eta_{i_j}^{q})^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon\nu_{i_j}^{q})^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)}\end{pmatrix}$$

$$= \left(\begin{array}{l} \sqrt[3]{\frac{1}{0.99} \times \left(1 - \left(\begin{array}{l} \left(2 - 0.99 - \left((1 - 0.99 \times (1 - 0.8^3))^{0.27/2} \times (1 - 0.99 \times (1 - 0.6^3))^{0.39/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times (1 - 0.8^3))^{0.27/2} \times (1 - 0.99 \times (1 - 0.9^3))^{0.34/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times (1 - 0.6^3))^{0.39/2} \times (1 - 0.99 \times (1 - 0.9^3))^{0.34/2}\right)\right) \end{array} \right)^{1/3} \right)}, \\ \sqrt[3]{1 - \frac{1}{0.99} \times \left(1 - \left(\begin{array}{l} \left(2 - 0.99 - \left((1 - 0.99 \times 0.3^3)^{0.27/2} \times (1 - 0.99 \times 0.7^3)^{0.39/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times 0.3^3)^{0.27/2} \times (1 - 0.99 \times 0.1^3)^{0.34/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times 0.7^3)^{0.39/2} \times (1 - 0.99 \times 0.1^3)^{0.34/2}\right)\right) \end{array} \right)^{1/3} \right)}, \\ \sqrt[3]{1 - \frac{1}{0.99} \times \left(1 - \left(\begin{array}{l} \left(2 - 0.99 - \left((1 - 0.99 \times 0.4^3)^{0.27/2} \times (1 - 0.99 \times 0.2^3)^{0.39/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times 0.4^3)^{0.27/2} \times (1 - 0.99 \times 0.5^3)^{0.34/2}\right)\right) \\ \times \left(2 - 0.99 - \left((1 - 0.99 \times 0.2^3)^{0.39/2} \times (1 - 0.99 \times 0.5^3)^{0.34/2}\right)\right) \end{array} \right)^{1/3} \right)} \end{array} \right)$$

$$= (0.915, 0.337, 0.274)$$

3.4TSFAdDHM Operator

The new TSFAdDHM operator based on the DHM operator and the AdOLs of TSFNs is put forward in this subsection.

Definition 10 Suppose a set of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$ (i=1,2,…,n), then the TSFAdDHM operator is proposed as (k=1, 2,…, n)

$$TSFAdDHM^{(k)}(\delta_1, \delta_2, \ldots, \delta_n) = \left(\underset{1 \le i_1 < \cdots < i_k \le n}{\otimes} \frac{\oplus_{j=1}^{k} \delta_{i_j}}{k} \right)^{1/C_n^k} \tag{14}$$

where C_n^k is a combinatorial number and (i_1, i_2,…, i_k) is a k-metric permutations of (1, 2,…,n).

Since the AdOLs for TSFNs, the below theorem can be drawn.

Theorem 7 Suppose a set of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$ (i=1,2,…,n), then the aggregation result according to Definition 10 can be denoted as (k=1, 2,…, n)

$$TSFAdDHM^{(k)}(\delta_1, \delta_2, \ldots, \delta_n) = \left(\begin{array}{l} \sqrt[q]{1 - \frac{1}{\varepsilon} \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(2 - \varepsilon - \prod_{j=1}^{k} (1 - \varepsilon \tau_{i_j}^q)^{1/k} \right) \right)^{1/C_n^k} \right)}, \\ \sqrt[q]{1 - \frac{1}{\varepsilon} \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(2 - \varepsilon - \prod_{j=1}^{k} (1 - \varepsilon \eta_{i_j}^q)^{1/k} \right) \right)^{1/C_n^k} \right)}, \\ \sqrt[q]{\frac{1}{\varepsilon} \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(2 - \varepsilon - \prod_{j=1}^{k} (1 - \varepsilon (1 - \nu_{i_j}^q))^{1/k} \right) \right)^{1/C_n^k} \right)} \end{array} \right) \tag{15}$$

Proof: Based on the AdOLs of TSFNs, we get

$$\oplus_{j=1}^{k}\delta_{i_j}=\left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\tau_{i_j}^{q})\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\eta_{i_j}^{q}))\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\nu_{i_j}^{q}))\right)}\right)$$

Then,

$$\frac{\oplus_{j=1}^{k}\delta_{i_j}}{k}=\left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon\tau_{i_j}^{q})^{1/k}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\eta_{i_j}^{q}))^{1/k}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{j=1}^{k}(1-\varepsilon(1-\nu_{i_j}^{q}))^{1/k}\right)}\right)$$

And

$$\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\frac{\oplus_{j=1}^{k}\delta_{i_j}}{k}=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\tau_{i_j}^{q})^{1/k}\right)\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\nu_{i_j}^{q}))^{1/k}\right)\right)}\end{array}\right)$$

Subsequently, we have

$$\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\frac{\oplus_{j=1}^{k}\delta_{i_j}}{k}\right)^{1/C_n^k}=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\tau_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{1/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\nu_{i_j}^{q}))^{1/k}\right)\right)^{1/C_n^k}\right)}\end{array}\right)$$

Therefore, Eq. (15) holds.

The TSFAdDHM operator features the Idempotency and Commutativity, which are demonstrated by thesame procedure as in the Theorems 3 and 4. Let an assortment of TSFNs be $\delta_i=(\tau_i, \eta_i, \nu_i)$, where i=1, 2,⋯, n, and these desirable properties are specifically described below.

Theorem 8 (Idempotency) If $\delta_i=\delta$ for all i, it means that all the TSFNs are the same, then

$$TSFAdDHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\delta \tag{16}$$

Theorem 9 (Commutativity) If $(\delta_1, \delta_2, \ldots,\delta_n)$ is arbitrary permuted as (σ1, σ2,⋯,σn), then

$$TSFAdHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=TSFAdHM^{(k)}(\sigma_1,\sigma_2,\ldots,\sigma_n) \tag{17}$$

Next, where the parameter k is assigned various values, this operator degenerates into these following particular cases in various decision-making settings.

Case 5 If the k-value is 1, The TSFAdDHM operator is degenerated to a TSFGM operator.

$$TSFAdDHM^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=$$

$$\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon\tau_{i_j}^{q})^{1/1}\right)\right)^{1/C_n^1}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon\eta_{i_j}^{q})^{1/1}\right)\right)^{1/C_n^1}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1 i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{1}(1-\varepsilon(1-\nu_{i_j}^{q}))^{1/1}\right)\right)^{1/C_n^1}\right)}\end{array}\right)$$

$$=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon(1-\tau_{i_1}^q)\right)^{1/n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon\eta_{i_1}^q\right)^{1/n}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{1\le i_1\le n}\left(1-\varepsilon\nu_{i_1}^q\right)^{1/n}\right)}\end{array}\right)=\frac{\otimes_{i=1}^n\delta_i}{n}=TSFGM(\delta_1,\delta_2,\ldots,\delta_n)$$

This operator is the TSF GM (Eq.(7)) in Remark 1.

Case 6 If the *k*-value is n, the TSFAdDHM operator is degenerated to a T-spherical fuzzy advanced averaging (TSFAdA) operator.

$$TSFAdDHM^{(n)}(\delta_1,\delta_2,\ldots,\delta_n)=$$

$$\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^n(1-\varepsilon\tau_{i_j}^q)^{1/n}\right)\right)^{1/C_n^n}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^n(1-\varepsilon\eta_{i_j}^q)^{1/n}\right)\right)^{1/C_n^n}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^n(1-\varepsilon(1-\nu_{i_j}^q))^{1/n}\right)\right)^{1/C_n^n}\right)}\end{array}\right)$$

$$=\left(\begin{array}{l}\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^n(1-\varepsilon\tau_i^q)^{1/n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^n(1-\varepsilon\eta_i^q)^{1/n}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^n(1-\varepsilon(1-\nu_i^q))^{1/n}\right)}\end{array}\right)=TSFAdA(\delta_1,\delta_2,\ldots,\delta_n)$$

Case 7 If the *k*-value is 1, the TSFAdDHM operator is degenerated to a q-rung orthopair fuzzy advanced geometric averaging (q-ROFAdGA) operator if the abstinence degree of TSFNs are all zero (i.e., η=0).

$$TSFAdDHM_{\eta=0}^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=$$

$$\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^1(1-\varepsilon\tau_{i_j}^q)^{1/1}\right)\right)^{1/C_n^1}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1 i_k\le n}\left(2-\varepsilon-\prod_{j=1}^1(1-\varepsilon(1-\nu_{i_j}^q))^{1/1}\right)\right)^{1/C_n^1}\right)}\right)$$

$$=\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon(1-\tau_{i_1}^q)\right)\right)^{1/n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon\nu_{i_1}^q\right)\right)^{1/n}\right)}\right)=\left(\otimes_{i=1}^n\delta_i\right)^{1/n}=q-ROFAdGA(\delta_1,\delta_2,\ldots,\delta_n)$$

Case 8 If both q and k are 1, the TSFAdDHM operator is degenerated to an intuitionistic fuzzy advanced geometric averaging (IFAdGA) operator if the abstinence degree of TSFNs is all zero (i.e., η=0).

$$TSFAdDHM_{q=1,\eta=0}^{(1)}(\delta_1,\delta_2,\ldots,\delta_n)=$$

$$\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(2-\varepsilon-\prod_{j=1}^1(1-\varepsilon\tau_{i_j})^{1/1}\right)\right)^{1/C_n^1}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1 i_k\le n}\left(2-\varepsilon-\prod_{j=1}^1(1-\varepsilon(1-\nu_{i_j}))^{1/1}\right)\right)^{1/C_n^1}\right)}\right)$$

$$=\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon(1-\tau_{i_1})\right)\right)^{1/n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1\le n}\left(1-\varepsilon\nu_{i_1}\right)\right)^{1/n}\right)}\right)=\left(\otimes_{i=1}^n\delta_i\right)^{1/n}=IFAdGA(\delta_1,\delta_2,\ldots,\delta_n)$$

Case 9 When q=1 and k=n, then TSFAdDHM is degraded to an IFAdA if the abstinence degree of TSFNs is all zero (i.e., η=0).

$$TSFAdDHM_{q=1,\eta=0}^{(n)}(\delta_1,\delta_2,\ldots,\delta_n)=$$

$$\left(\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{n}(1-\varepsilon\tau_{i_j})^{1/n}\right)\right)^{1/C_n^n}\right)},\sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{n}(1-\varepsilon(1-\nu_{i_j}))^{1/n}\right)\right)^{1/C_n^n}\right)}\right)$$

$$=\left(\sqrt[q]{\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon\tau_i)^{1/n}\right)},\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\prod_{i=1}^{n}(1-\varepsilon(1-\nu_i))^{1/n}\right)}\right)=\frac{\oplus_{i=1}^{n}\delta_i}{n}=IFAdA(\delta_1,\delta_2,\ldots,\delta_n)$$

Clearly, the IFAdA operator is a particular form of the AIVIFA operator proposed by Kumar and Chen [35].

3.5 TSFAdWDHM Operator

Similarly, the input arguments are not assigned importance in the TSFAdDHM operator. To eliminate the limitations of the TSFAdDHM operator, its weighted form (TSFAdWDHM) is proposed.

Definition 11 Suppose an assortment of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$, the weight of δ_i is w_i, fulfilling $0\le w_i\le 1$ and $\sum_{i=1}^{n} w_i=1$, then the TSFAdWDHM is introduced as (k=1,2,…,n)

$$TSFAdWDHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\left(\underset{1\le i_1<\cdots<i_k\le n}{\otimes}\frac{\oplus_{j=1}^{k}(w_{i_j}\otimes\delta_{i_j})}{k}\right)^{1/C_n^k} \tag{18}$$

where C_n^k is a combinatorial number and $(i_1, i_2,\ldots, i_k)$ is a k-metric permutations of (1,2,…,n).

Theorem 10 Suppose a collection of TSFNs is $\delta_i=(\tau_i, \eta_i, \nu_i)$, the weight of δ_i is w_i, fulfilling $0\le w_i\le 1$ and $\sum_{i=1}^{n} w_i=1$, then the result shape of TSFAdWDHM can be expressed as (k=1,2,…,n)

$$TSFAdWDHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\tau_{i_j}^{q})^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon\eta_{i_j}^{q})^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_k\le n}\left(2-\varepsilon-\prod_{j=1}^{k}(1-\varepsilon(1-\nu_{i_j}^{q}))^{w_{i_j}/k}\right)\right)^{1/C_n^k}\right)}\end{array}\right) \tag{19}$$

It proves the same process as the Theorem 4, omitted.

Example 4 We use the three TSFNs in Example 3 and the appropriate weight vector, and then fuse them employing the TSFAdWDHM to obtain a composite value, which is calculated in detail as follows (q=3, ε=0.99):

$$TSFAdWDHM^{(2)}(\delta_1,\delta_2,\delta_3)=\left(\begin{array}{l}\sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon\tau_{i_j}^{q})^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)},\\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon\eta_{i_j}^{q})^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)},\\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod_{1\le i_1<\cdots<i_2\le n}\left(2-\varepsilon-\prod_{j=1}^{2}(1-\varepsilon(1-\nu_{i_j}^{q}))^{w_{i_j}/2}\right)\right)^{1/C_3^2}\right)}\end{array}\right)$$

$$=\left(\begin{array}{l}\sqrt[3]{1-\frac{1}{0.99}\times\left(1-\left(\begin{array}{l}\left(2-0.99-\left((1-0.99\times0.8^3)^{0.27/2}\times(1-0.99\times0.6^3)^{0.39/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times0.8^3)^{0.27/2}\times(1-0.99\times0.9^3)^{0.34/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times0.6^3)^{0.39/2}\times(1-0.99\times0.9^3)^{0.34/2}\right)\right)\end{array}\right)^{1/3}\right)},\\ \sqrt[3]{1-\frac{1}{0.99}\times\left(1-\left(\begin{array}{l}\left(2-0.99-\left((1-0.99\times0.3^3)^{0.27/2}\times(1-0.99\times0.7^3)^{0.39/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times0.3^3)^{0.27/2}\times(1-0.99\times0.1^3)^{0.34/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times0.7^3)^{0.39/2}\times(1-0.99\times0.1^3)^{0.34/2}\right)\right)\end{array}\right)^{1/3}\right)},\\ \sqrt[3]{\frac{1}{0.99}\times\left(1-\left(\begin{array}{l}\left(2-0.99-\left((1-0.99\times(1-0.4^3))^{0.27/2}\times(1-0.99\times(1-0.2^3))^{0.39/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times(1-0.4^3))^{0.27/2}\times(1-0.99\times(1-0.5^3))^{0.34/2}\right)\right)\\ \times\left(2-0.99-\left((1-0.99\times(1-0.2^3))^{0.39/2}\times(1-0.99\times(1-0.5^3))^{0.34/2}\right)\right)\end{array}\right)^{1/3}\right)}\end{array}\right)$$

$$=(0.589,0.337,0.725)$$

4. A Proposed Model Based on the Proposed Operators

The developed TSFAdWHM and TSFAdWDHM operators are employed to solve the MAGDM problem with the TSF information in this section.

Let H={$h_1,h_2,\cdots,h_m$} be an assortment of alternatives, A={$a_1,a_2,\cdots,a_n$} be a family containing n attributes, and E={$e_1,e_2,\cdots,e_p$} be a group of decision-makers. The corresponding attribute and decision-maker weight vectors are denoted w=$(w_1,w_2,\cdots,w_n)^T$ and $\omega=(\omega_1,\omega_2,\ldots,\omega_p)^T$, respectively, and they satisfy $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$ and $\omega_t\in[0,1]$ and $\sum_{t=1}^{p}\omega_t=1$. Some decision-makers E={e_1, e_2, $\cdots$, e_p} are invited to use TSFN to express his or her judgment about alternative h_i regarding attribute a_j, it is denoted as $\delta_{ij}^t=(\tau_{ij}^t,\eta_{ij}^t,\nu_{ij}^t)$ (i=1,2,$\cdots$,m; j=1,2,$\cdots$,n; t=1,2,$\cdots$,p). Thus, each individual TSF decision-making matrix can be obtained, that is noted as $D^t=\left[\delta_{ij}^t\right]_{m\times n}$.

We give the detailed algorithm for solving the TSF MAGDM problems as follows:

Step 1 In many practical decision problems, some attributes are usually categorized into benefit and cost types. For this reason, the initial individual TSF decision matrix is normalized to remove the effect of various attribute types. Then the standardized initial individual TSF decision matrix $S^t=\left[\sigma_{ij}^t\right]_{m\times n}$ is obtained based on Eq. (20).

$$\sigma_{ij}^{t} = \begin{cases} \delta_{ij}^{t} = (\tau_{ij}^{t}, \eta_{ij}^{t}, \nu_{ij}^{t}) & j \in J_1 \\ (\delta_{ij}^{t})^{c} = (\nu_{ij}^{t}, \eta_{ij}^{t}, \tau_{ij}^{t}) & j \in J_2 \end{cases} \tag{20}$$

where J_1 and J_2 denote benefit and cost attribute types, respectively. $(\delta_{ij}^{t})^{c}$ is the complement of δ_{ij}^{t}.

Step 2 The normalized individual TSF decision matrices S^t (t=1, 2,..., p) from the various decision-makers are fused by utilizing the TSFAdWHM operator (Eq. (21)) or the TSFAdWDHM operator (Eq. (22)). The TSF group decision-making matrix is obtained, i.e., $G=[g_{ij}]_{m\times n}$.

$$g_{ij} = TSFAdWHM^{(k)}(\sigma_{ij}^{1}, \sigma_{ij}^{2}, \ldots, \sigma_{ij}^{p}) = \left(\begin{array}{l} \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{p}(1-\varepsilon(1-(\tau_{ij}^{t_l})^{q}))^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)}, \\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{p}(1-\varepsilon(\eta_{ij}^{t_l})^{q})^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)}, \\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{p}(1-\varepsilon(\nu_{ij}^{t_l})^{q})^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)} \end{array} \right) \tag{21}$$

or

$$g_{ij} = TSFAdWDHM^{(k)}(\sigma_{ij}^{1}, \sigma_{ij}^{2}, \ldots, \sigma_{ij}^{p}) = \left(\begin{array}{l} \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon(\tau_{ij}^{t_l})^{q})^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)}, \\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon(\eta_{ij}^{t_l})^{q})^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)}, \\ \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le t_1<\cdots<t_k\le p}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon(1-(\nu_{ij}^{t_l})^{q}))^{\omega_{t_l}/k}\right)\right)^{1/C_p^k}\right)} \end{array} \right) \tag{22}$$

Step 3 The evaluated values under all attributes are aggregated into a comprehensive evaluated value applying the TSFAdWHM operator (Eq. (23)) or the TSFAdWDHM operator (Eq. (24)). The comprehensive value of alternative h_i is noted as x_i.

$$x_i = TSFAdWHM^{(k)}(g_{i1}, g_{i2}, \ldots, g_{in}) = \left(\begin{array}{l} \sqrt[q]{\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le j_1<\cdots<j_k\le n}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon(1-\tau_{ij_l}^{q}))^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)}, \\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le j_1<\cdots<j_k\le n}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon\eta_{ij_l}^{q})^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)}, \\ \sqrt[q]{1-\frac{1}{\varepsilon}\left(1-\left(\prod\limits_{1\le j_1<\cdots<j_k\le n}\left(2-\varepsilon-\prod_{l=1}^{k}(1-\varepsilon\nu_{ij_l}^{q})^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)} \end{array} \right) \tag{23}$$

or

$$x_i = TSFAdWDHM^{(k)}(g_{i1}, g_{i2}, \ldots, g_{in}) = \left(\begin{array}{l} \sqrt[q]{1 - \frac{1}{\varepsilon}\left(1 - \left(\prod_{1 \le j_1 < \cdots < j_k \le n} \left(2 - \varepsilon - \prod_{l=1}^{k} (1 - \varepsilon \tau_{ij_l}^{q})^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)}, \\ \sqrt[q]{1 - \frac{1}{\varepsilon}\left(1 - \left(\prod_{1 \le j_1 < \cdots < j_k \le n} \left(2 - \varepsilon - \prod_{l=1}^{k} (1 - \varepsilon \eta_{ij_l}^{q})^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)}, \\ \sqrt[q]{\frac{1}{\varepsilon}\left(1 - \left(\prod_{1 \le j_1 < \cdots < j_k \le n} \left(2 - \varepsilon - \prod_{l=1}^{k} (1 - \varepsilon(1 - \nu_{ij_l}^{q}))^{w_{j_l}/k}\right)\right)^{1/C_n^k}\right)} \end{array} \right) \quad (24)$$

Step 4 The comprehensive values of the options are converted into score values by Eq.(2).

Step 5 Ranking alternatives and selecting the best option, i.e., the larger the score value, the better the option is. The flowchart for the proposed approach is shown in Fig.2.

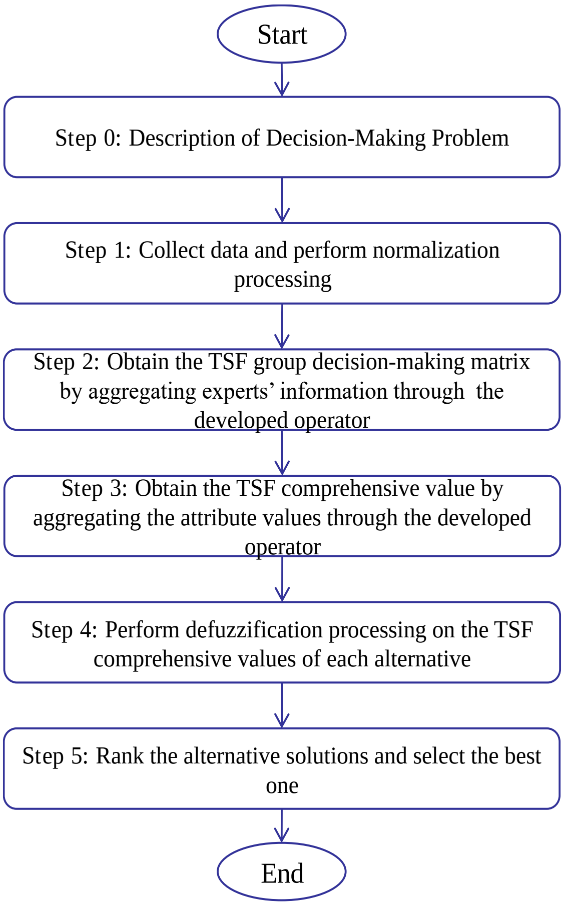


Fig. 2 the flowchart of the proposed approach

5 Numerical Examples

The practicability of the methodology as mentioned above is examined through a real life case study on "Selection of a project by an architectural firm"(cited from Hussain et al. [30]) in this section.

Example 5 An architectural firm wants to assess performance of their projects and identify the best one(s). The company has organized a panel of decision-makers E={e_1,e_2,e_3} who will perform the evaluation. Their weight vector was given as ω=(0.2,0.1,0.7)T depending on the experts. The decision-makers assessed the importance rating based on some performance indicators (attributes), i.e., the rate of return (a_1), the time to completion (a_2), the total investment cost (a_3) and the customer feedback (a_4). Their weight vector was given as w=(0.2,0.3,0.3,0.2)^T. The projects were initially assessed and screened, and there are four projects as a collection of alternatives, i.e., H={h_1, h_2, h_3, h_4}. The initial decision-making matrices received from the decision-makers were expressed as TSFNs in Table 2 below. (k=2, q=3)

Table 2 TSFNs provided by the three decision-making

Decision-makers	Alternatives	a_1	a_2	a_3	a_4
e_1	h_1	(.25, .54, .45)	(.42, .31, .48)	(.35, .29, .65)	(.32, .33,.48)
	h_2	(.61, .54, .43)	(.43, .56, .36)	(.45, .67, .45)	(.54, .43,.63)
	h_3	(.47, .37, .39)	(.55, .39, .38)	(.47, .46, .62)	(.38, .25,.48)
	h_4	(.52, .51, .44)	(.27, .25, .33)	(.43, .53, .43)	(.27, .34,.32)
e_2	h_1	(.39, .54, .37)	(.38, .28, .26)	(.29, .42, .44)	(.38, .55,.32)
	h_2	(.35, .34, .19)	(.39, .28, .43)	(.27, .27, .37)	(.27, .61,.71)
	h_3	(.49, .43, .77)	(.65, .62, .33)	(.43, .31, .26)	(.28, .39,.41)
	h_4	(.42, .11, .29)	(.34, .35, .53)	(.25, .63,.54)	(.27,.43,.67)
e_3	h_1	(.26, .44, .38)	(.35, .38, .53)	(.53, .27, .49)	(.43,.43,.44)
	h_2	(.39, .36, .49)	(.41, .25, .55)	(.47, .47, .59)	(.54,.39,.66)
	h_3	(.69, .36, .37)	(.49, .38, .49)	(.38, .44, .41)	(.33,.47,.43)
	h_4	(.52, .19, .37)	(.54, .37, .48)	(.65, .66, .47)	(.47,.49,.59)

5.1 The Application of proposed method

(1) Using the TSFAdWHM operator

Step 1 As a3 is a cost-based attribute, the TSFNs under a_3 are normalized using Eq. (18). The standardized evaluation matrix can be obtained as shown in Table 3.

Step 2 Use the Eq. (19) to fuse the evaluation information of each decision-maker. We can get a group decision-making matrix with relevant data listed in Table 4.

Table 3 the normalized TSF evaluation matrix

Decision-makers	Alternatives	a_1	a_2	a_3	a_4
e_1	h_1	(.25, .54, .45)	(.42, .31, .48)	(.65, .29, .35)	(.32, .33, .48)
	h_2	(.61, .54, .43)	(.43, .56, .36)	(.45, .67, .45)	(.54, .43, .63)
	h_3	(.47, .37, .39)	(.55, .39, .38)	(.62, .46, .47)	(.38, .25, .48)
	h_4	(.52, .51, .44)	(.27, .25, .33)	(.43, .53, .43)	(.27, .34, .32)
e_2	h_1	(.39, .54, .37)	(.38, .28, .26)	(.44, .42, .29)	(.38, .55, .32)
	h_2	(.35, .34, .19)	(.39, .28, .43)	(.37, .27, .27)	(.27, .61, .71)
	h_3	(.49, .43, .77)	(.65, .62, .33)	(.26, .31, .43)	(.28, .39, .41)
	h_4	(.42, .11, .29)	(.34, .35, .53)	(.54, .63, .25)	(.27, .43, .67)
e_3	h_1	(.26, .44, .38)	(.35, .38, .53)	(.49, .27, .53)	(.43, .43, .44)
	h_2	(.39, .36, .49)	(.41, .25, .55)	(.59, .47, .47)	(.54, .39, .66)
	h_3	(.69, .36, .37)	(.49, .38, .49)	(.41, .44, .38)	(.33, .47, .43)
	h_4	(.52, .19, .37)	(.54, .37,.48)	(.47, .66, .65)	(.47, .49, .59)

Table 4 TSF group decision matrix

Alternatives	a_1	a_2	a_3	a_4
h_1	(.720, .331, .272)	(.767, .242, .337)	(.834, .206, .316)	(.769, .295, .300)
h_2	(.801, .284, .308)	(.783, .251, .335)	(.824, .360, .305)	(.816, .302, .460)
h_3	(.859, .255, .326)	(.837, .294, .304)	(.797, .296, .282)	(.747, .285, .302)
h_4	(.822, .215, .261)	(.785, .236, .313)	(.807, .431, .380)	(.762, .307, .383)

Step 3 The Eq. (21) is employed to generate the aggregated values of alternatives, and the aggregation results are listed as following: x_1=(0.940, 0.169, 0.197), x_2=(0.948, 0.193, 0.225), x_3=(0.950, 0.180, 0.191), x_4=(0.945, 0.203, 0.216).

Step 4 From the Eq. (2), the score values are further computed for the above mentioned options, i.e., 0.908, 0.917, 0.922, 0.912.

Step 5 From the results of the previous step, we rank these alternatives as $h_3 \succ h_2 \succ h_4 \succ h_1$.

Thus, h_3 is the optimal project.

(2) The decision-making process based on the TSFAdWDHM operator

Step 1 We normalized using the Eq. (18) to obtain Table 3.

Step 2 The Eq. (20) is employed to fuse the assessment information from distinct experts to produce the group decision matrix with TSFNs, Table 5 displays these data.

Step 3 The combined evaluated value of every option is calculated utilizing the Eq. (22) and we can obtain x_1=(0.265, 0.246, 0.944), x_2=(0.305, 0.280, 0.950), x_3=(0.321, 0.262, 0.943), x_4=(0.293, 0.293, 0.947).

Table 5 TSF group decision matrix

Alternatives	a_1	a_2	a_3	a_4
h_1	(.192, .331, .778)	(.255, .242, .806)	(.369, .206, .790)	(.273, .295, .790)
h_2	(.315, .284, .784)	(.282, .251, .807)	(.362, .360, .789)	(.355, .302, .885)
h_3	(.419, .255, .794)	(.367, .294, .788)	(.322, .296, .786)	(.232, .285, .796)
h_4	(.348, .215, .766)	(.314, .236, .793)	(.323, .431, .826)	(.278, .307, .825)

Step 4 Further, the corresponding score values of these alternatives are 0.081, 0.074, 0.088, and 0.075 from Eq.(2).

Step 5 Based on the results of Step 4, the options are prioritized as $h_3 \succ h_1 \succ h_4 \succ h_2$.

Hence, alternative h_3 is the best project.

From the above, we find that the optimal results obtained by the TSFAdWHM and TSFAdWDHM in Example 5 are the same. Therefore, we can identify h3 as the optimal project option.

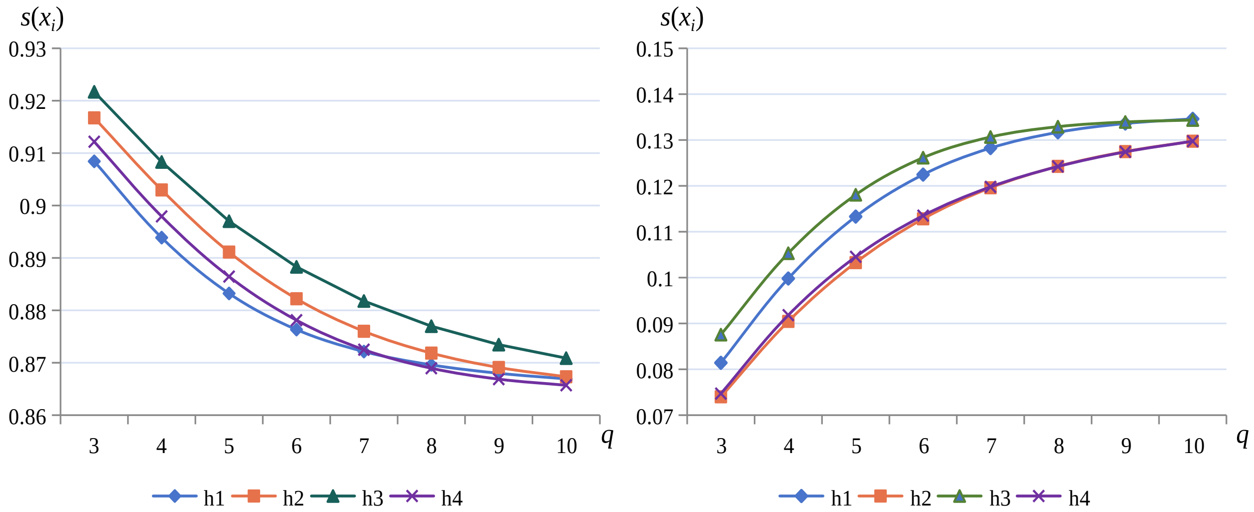
5.2 Sensitivity Analysis on q, ε and k

The proposed methodology is performed a sensitivity analysis on q, ε and k to detect the effect of these parameters on the results of the alternatives ranking.

In the case of ε=0.99 and k=2, the q takes values from 3 to 10 to perform the sensitivity analysis of the proposed method, and Table 6 and Fig. 2 reflect the variation in the outcomes originating from the two algorithms.

Table 6The results of proposed methods based on AOs regarding q

q	TSFAdWHM-based		TSFAdWDHM-based	
	Score values	Ranking	Score values	Ranking
3	.9084, .9167, .9217, .9122	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0814, .0740, .0876, .0747	$h_3 \succ h_1 \succ h_4 \succ h_2$
4	.8939, .9030, .9083, .8979	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0998, .0904, .1053, .0918	$h_3 \succ h_1 \succ h_4 \succ h_2$
5	.8832, .8911, .8970, .8864	$h_3 \succ h_2 \succ h_4 \succ h_1$	.1133, .1033, .1181, .1045	$h_3 \succ h_1 \succ h_4 \succ h_2$
6	.8763, .8822, .8883, .8781	$h_3 \succ h_2 \succ h_4 \succ h_1$	.1225, .1128, .1261, .1135	$h_3 \succ h_1 \succ h_4 \succ h_2$
7	.8721, .8760, .8818, .8725	$h_3 \succ h_2 \succ h_4 \succ h_1$	.1282, .1196, .1307, .1198	$h_3 \succ h_1 \succ h_4 \succ h_2$
8	.8696, .8718, .8770, .8690	$h_3 \succ h_2 \succ h_1 \succ h_4$	.1317, .1243, .1329, .1242	$h_3 \succ h_1 \succ h_2 \succ h_4$
9	.8680, .8691, .8735, .8669	$h_3 \succ h_2 \succ h_1 \succ h_4$	.1336, .1275, .1339, .1274	$h_3 \succ h_1 \succ h_2 \succ h_4$
10	.8669, .8673, .8709, .8657	$h_3 \succ h_2 \succ h_1 \succ h_4$	.1346, .1297, .1344, .1297	$h_3 \succ h_1 \succ h_2 \approx h_4$

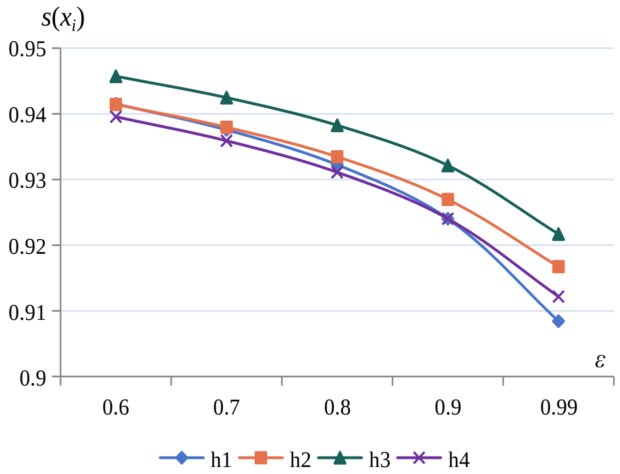


a. TSFAdWHM operator *b.* TSFAdWDHM operator

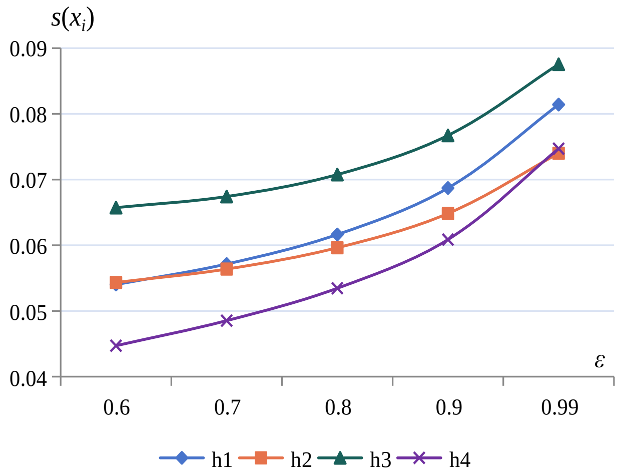
Fig.3 the change in score values with q

The score values obtained from the methods based on the TSFAdWHM and TSFAdWDHM operators have been listed in Table 6, as have the corresponding rankings for each alternative. For the TSFAdWHM-based model, the q takes on various values from 3 to 10, and the ordering of the alternatives changes from $h_3 \succ h_2 \succ h_4 \succ h_1$ to $h_3 \succ h_2 \succ h_1 \succ h_4$.From Fig.3a, the changes in the score function values of the alternatives show a decreasing trend, and the difference in their score function values gradually becomes smaller as the parameter q value increases. For the TSFAdWDHM-based model, the prioritization of options for q-value changes from $h_3 \succ h_1 \succ h_4 \succ h_2$ to $h_3 \succ h_1 \succ h_2 \succ h_4$. Fig.3b reveals that the values of score function for the alternatives increases gradually as q increase, while the difference between their score function values gradually becomes smaller. Hence, the desirable q value can be picked by the decision-maker depending on the actual decision-making scenario, while providing a larger and freer decision-making space for effectively processing evaluation information.

In the case of q=3and k=2, we perform a sensitivity analysis of the proposed methods regarding the parameter ε taking different values in the range [0.6,0.99]. We obtain the results for the methods based on TSFAdWHM and TSFAdWDHM operators regarding different values of ε, respectively. The sorting changes are presented in Fig. 3, as well as the results are listed in Table 7.



a. TSFAdWHM operator



b. TSFAdWDHM operator

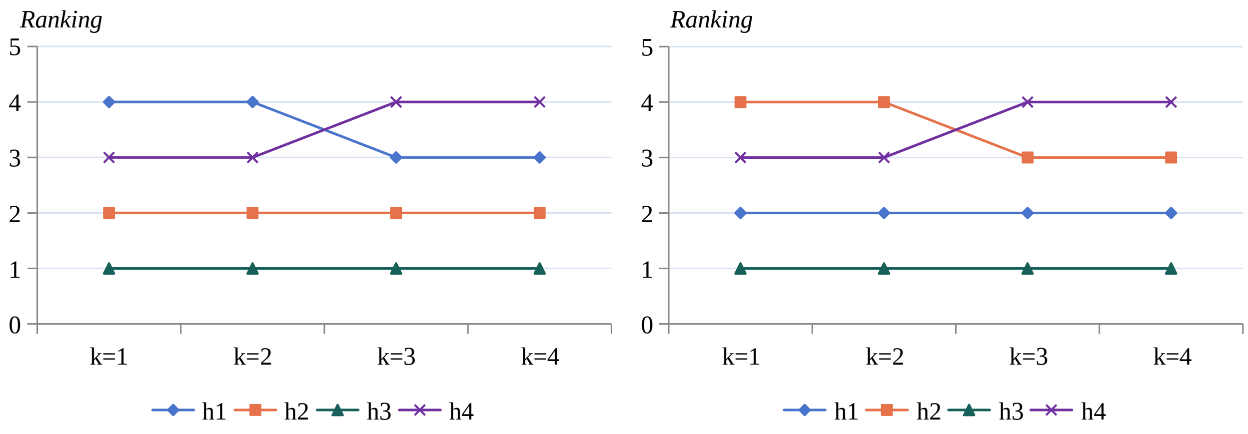
Fig.4 Variation of score values with ε

Table 7 The results of approach with regard to ε

ε	TSFAdWHM-based		TSFAdWDHM-based	
	Score values	Ranking	Score values	Ranking
0.99	.9084, .9167, .9217, .9122	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0814, .0740, .0876, .0747	$h_3 \succ h_1 \succ h_4 \succ h_2$
0.9	.9240, .9270, .9321, .9243	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0687, .0648, .0767, .0609	$h_3 \succ h_1 \succ h_2 \succ h_4$
0.8	.9322, .9335, .9383, .9311	$h_3 \succ h_2 \succ h_1 \succ h_4$	.0616, .0596, .0708, .0535	$h_3 \succ h_1 \succ h_2 \succ h_4$
0.7	.9376, .9380, .9425, .9259	$h_3 \succ h_2 \succ h_1 \succ h_4$	.0571, .0564, .0674, .0485	$h_3 \succ h_1 \succ h_2 \succ h_4$
0.6	.9415, .9414, .9457, .9396	$h_3 \succ h_2 \succ h_1 \succ h_4$	.0540, .0543, .0657, .0447	$h_3 \succ h_1 \succ h_2 \succ h_4$

From the results of Table 7, the best alternative is consistently h_3, and the priority of other alternatives has changed. This result can also be found in Fig. 4. As the value of parameter ε varies from 0.6 to 0.99, the trend of the score function values resulting from the TSFAdWHM-based and TSFAdWDHM-based approaches in Fig. 4 is opposite, i.e., the method based on TSFAdWHM operator gradually decreases (Fig. 4a), while the method based on TSFAdWDHM operator gradually increases (Fig. 4b). As a result, we believe that the ε can express the attitude of the decision-makers. Specifically, for the TSFAdWHM-based method if there are more optimistic decision-makers, the parameter ε can take a smaller value; if there are more pessimists, the parameter ε can take a larger value. For the TSFAdWDHM-based method, the ε takes the opposite value.

In Example 5, the k can be taken as 1, 2, 3 and 4 to reflect different association scenarios between attributes. Next, a sensitivity analysis is conducted on the decision results regarding parameter k(when q=3, ε=0.99).As a result, we obtain the prioritization of the options for each of the TSFAdWHM-based and TSFAdWDHM-based methods, respectively. The sorting changes are described in Fig.5, as well as the outcomes are listed in Table 8.



a. TSFAdWHM operator *b*. TSFAdWDHM operator

Fig. 5 Ranking of alternatives on k

Table 8 The results of proposed method based on AOs regarding k

k	TSFAdWHM-based		TSFAdWDHM-based	
	Score values	Ranking	Score values	Ranking
1	.9093, .9183, .9237, .9142	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0793, .0683, .0809, .0740	$h_3 \succ h_1 \succ h_4 \succ h_2$
2	.9084, .9167, .9217, .9122	$h_3 \succ h_2 \succ h_4 \succ h_1$	.0814, .0740, .0876, .0747	$h_3 \succ h_1 \succ h_4 \succ h_2$
3	.9376, .9380, .9425, .9361	$h_3 \succ h_2 \succ h_1 \succ h_4$	.0542, .0513, .0573, .0503	$h_3 \succ h_1 \succ h_2 \succ h_4$
4	.9375, .9378, .9423, .9358	$h_3 \succ h_2 \succ h_1 \succ h_4$	.0543, .0515, .0575, .0503	$h_3 \succ h_1 \succ h_2 \succ h_4$

From Table 8 and Fig.5, when k=1, k=2 the alternatives are ranked the same, while when k=3 and k=4 the alternatives are ranked the same. Although the attributes are independent of each other when k=1 and k=4, there are three attributes' correlations when k=3, which is more than the number of associations when k=2.This also shows the dexterity of the developed aggregation operators for the information fusion and that they can handle the MAGDM problems with interrelationship between attributes with different parameter k. In addition, we find that the larger the parameter k value, the smaller the difference between the aggregation operators' results. Therefore, decision-makers can choose the right k value according to their preferences and the requirements of practical decision-making.

5.3 Comparative analysis

Some existing aggregation operator-based methodologies are applied to resolve Example 5to test the performance and superiority of the designed methodologies. Considering that the developed methodologies based on TSFAdWHM and TSFAdWDHM operators can portray the correlation among multiple attributes and overcome the counter-intuitive features in the TSF environment. The following methods focus on three aspects of validity, correlation and counter-intuition to solve Example 5 and the superiority of the presented methods are elaborated by comparing and analyzing the outcomes, which are illustrated in Table 9.

Table 9 comparison of various operators for Example 5

Operators	Score values of h_i(i=1,2,3,4)	Ranking
q-ROFIWHM[42]	Cannot be calculated	No
PFNWBM[43]	Cannot be calculated	No
SFDWHeM[44]	Cannot be calculated	No
TSFWA [21]	.4709, .4716, .4988, .4708	$h_3 \succ h_2 \succ h_1 \succ h_4$
TSFWG [21]	.4614, .4576, .4854, .4648	$h_3 \succ h_4 \succ h_1 \succ h_2$
TSFWAI[24]	.4589, .4401, .4881, .4122	$h_3 \succ h_1 \succ h_2 \succ h_4$

TSFWGI [24]	.4574, .4434, .4890, .4166	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFAAWHM[29]	.7246, .7393, .7536, .7267	$h_3 \succ h_2 \succ h_4 \succ h_1$
TSFWGMSM[45]	.4594, .4572, .4945, .4603	$h_3 \succ h_4 \succ h_1 \succ h_2$
TSFWDBM[32]	.3795, .3568, .3988, .3755	$h_3 \succ h_1 \succ h_4 \succ h_2$
TSFIPWHeM[25]	.4908, .4805, .5101, .4852	$h_3 \succ h_1 \succ h_4 \succ h_2$
TSFWIPBM[46]	.4630, .4180, .5071, .4285	$h_3 \succ h_1 \succ h_4 \succ h_2$
TSFAdWHM(k=1)	.9093, .9183, .9237, .9142	$h_3 \succ h_2 \succ h_4 \succ h_1$
TSFAdWHM(k=2)	.9084, .9167, .9217, .9122	$h_3 \succ h_2 \succ h_4 \succ h_1$
TSFAdWDHM(k=1)	.0793, .0683, .0809, .0740	$h_3 \succ h_1 \succ h_4 \succ h_2$
TSFAdWDHM(k=2)	.0814, .0740, .0876, .0747	$h_3 \succ h_1 \succ h_4 \succ h_2$

In solving the MAGDM problem of Example 5 using the approaches presented by Xing et al. [42], Ates and Akay [43], and Zhang et al. [44], we find that these methods could not be used. This result is attributed to the fact that the TSFNs in Table 2 contain evaluation information that cannot be handled by q-rung orthopair fuzzy set, picture fuzzy set and spherical fuzzy set. This can show that the TSFS in this paper possesses the advantage of being able to handle vague, ambiguous and indeterminate complex decision-making issues.

We utilize the methods of Ullah et al. [21] based on TSFWA and TSFWG operators to handle Example 5, which is presented in Table 9. The method based on TSFAdWHM and TSFAdWDHM operators (k=1) can obtain the same optimal alternative (h3), which shows that no interrelationship is considered between the attributes. This fact indicates the agility and validity of the designed method. However, the method based on the TSFWA and TSFWG operators is fundamentally different from the TSFAdWHM-based and TSFAdWDHM-based methods (k=2).

The TSFWAI and TSFWGI operator-based methods designed by Ju et al. [24] to settle Example 5 and the obtained results are listed in Table 9. We find that both the TSFAdWDHM-based method (k = 4) and the TSFWAI-based method obtain alternatives ordering as $h_3 \succ h_1 \succ h_2 \succ h_4$. The cause for this is that the TSFAdWDHM operator degenerates to the TSFAdWA operator when k = 4 and both AdOLs and IOLs possess the effect of eliminating counter-intuition. Our proposed TSFAdWHM and TSFAdWDHM operators not only capture the association relationships among attributes, but also avoid the counter-intuitive situation that occurs when the value of a certain membership function in TSFN is zero. Therefore, the results show that the developed method is more rational than that of Ju et al. [24].

We use the TSFAAWHM-based method [29], the TSFWDBM-based method [32] and the TSFWGMSM-based method [45] to address Example 5. We find from Table 9 that the TSFAdWHM-based method (k=2) has the same result as the TSFAAWHM-based method [29] (k=2), i.e., $h_3 \succ h_2 \succ h_4 \succ h_1$. Meanwhile, both the TSFAdWDHM-based method(k=2) proposed and the TSFWDBM-based method [32] result in $h_3 \succ h_1 \succ h_4 \succ h_2$, whereas the TSFWGMSM operator-based model raised by Liu et al. [45] was able to identify h_3 as the optimal option, and the ranking of the remaining alternatives does not agree with the method in this article. In terms of operational rules, the TSFAAWHM, TSFWGMSM and TSFWDBM operators are based on the AAOL, AOL and DOL of the TSFNs, respectively. Although AAOLs and DOLs are more computationally complex than AdOLs, all three of them have high decision-making flexibility. However, these methods based on TSFAAWHM, TSFWGMSM and TSFWDBM operators do not have the ability to eliminate counter-intuition. Thus, in comparison, the developed method has the advantage of being counter-intuition in the decision-making process and the results derived are more rational and reliable.

The TSFIPWHeM-based method [25] and the TSFWIPBM-based method [46] are applied in Example 5, and Table 9 lists the results of the calculations. As can be seen in Table 9, the results derived from these two approaches are consistent with the TSFAdWDHM-based method (k=2) in this paper, i.e., $h_3 \succ h_1 \succ h_4 \succ h_2$.The TSFIPWHeM and TSFWIPBM operator-based models have the ability to both portray the association between attributes and realize the interactions between ternary membership functions in TSFNs. Although all these aggregation operators contain parameters to achieve decision-making flexibility, the proposed methods are more reasonable and advantageous in terms of function structure and computation.

As mentioned above, both newly developed and current methods are capable of obtaining optimal alternative h3, which can show that the proposed methods are able to settle the TSF MAGDM issues in a feasible and effective way. To further clarify the main benefits of the presented approach in this article, several data in Example 5 are adapted, and the methods are then analyzed comparatively.

Example 6 The ternary membership functions for each of the TSFNs in Table 2 are all non-zero values in Example 5. We select five TSFNs in Table 2 to change the ternary membership functions to zero value respectively, i.e., the value of $\delta_{42}{}^{1}$ changes from (0.27, 0.25, 0.33) to (0.00, 0.25, 0.33); the value of $\delta_{11}{}^{2}$ changes from (0.39, 0.54, 0.37) to (0.39, 0.54, 0.00); the value of $\delta_{23}{}^{2}$ changes from (0.37, 0.27, 0.27) to (0.37, 0.27, 0.00); the value of $\delta_{14}{}^{3}$ changes from (0.43, 0.43, 0.44) to (0.43, 0.00, 0.44); and the value of $\delta_{22}{}^{3}$ changes from (0.41, 0.25, 0.55) to (0.41, 0.00, 0.55). Then, we address the TSF MAGDM issue by applying the existing and the proposed methods respectively. Table 10 shows the results.

Table 10 Results of different approaches

Operators	Score values of h_i(i=1,2,3,4)	Ranking
TSFWA[21]	.5394, .5602, .4988, .4703	$h_2 \succ h_1 \succ h_3 \succ h_4$
TSFWG[21]	.4910, .4982, .4854, .4644	$h_2 \succ h_1 \succ h_3 \succ h_4$
TSFWAI[24]	.4647, .4416, .4881, .4115	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFWGI[24]	.4634 ,.4453, .4890, .4162	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFAAWHM[29]	.7263, .7398, .7536, N/A	No
TSFWGMSM[45]	.4809, .4753, .4945, .4599	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFWDBM[32]	N/A, N/A, .3988, .3753	No
TSFIPWHeM[25]	.4955, .4896, .5101, .4849	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFWIPBM[46]	.4815, .4348, .5071, .4277	$h_3 \succ h_1 \succ h_2 \succ h_4$
TSFAdWHM(k=2)	.9089, .9169, .9217, .9100	$h_3 \succ h_2 \succ h_4 \succ h_1$
TSFAdWDHM(k=2)	.0818, .0759, .0876, .0744	$h_3 \succ h_1 \succ h_2 \succ h_4$

From Table 10, the best option achieved by the TSFWA and TSFWG [21] has changed from h3 to h_2, while the results obtained by the proposed method have not changed significantly. The ranking achieved by the TSFAdWHM-based method is the same as the original one, and the ranking of h_3 and h_1 obtained by the TSFAdWDHM-based method is unchanged but the ranking of h_2 and h_4 is slightly changed. The methods based on the TSFWAI [24], TSFWGI [24], TSFWGMSM [45], TSFIPWHeM [25] and TSFWIPBM [46] are able to produce outcomes consistent with the TSFAdWDHM-based model, i.e., $h_3 \succ h_1 \succ h_2 \succ h_4$. However, the methods based on the TSFAAWHM [29] and TSFWDBM [32] do not yield a prioritization of alternatives. The rationale is that the AAOLs and DOLs in the TSFAAWHM [29] and TSFWDBM [32] fail during the TSFNs fusion process. Their expressions are shown in Eqs. (25) and (26), respectively. For example, $\delta_{42}{}^{1}$=(0.00,0.25,0.33), and when applying Eq. (25), the truth of the logarithm is zero, which results in the MD of Eq. (25) not being able to be found, which in turn prevents determining the ordering of alternative h_4. $\delta_{11}{}^2$=(0.39,0.54,0.00) and $\delta_{14}{}^3$=(0.43,0.00,0.44), when applying Eq. (26), both degrees of abstinence and non-membership, which are the denominators, are zero, which leads to the unavailability of the aggregated TSFNs for alternatives h_1 and h_2.

$$TSFAAWHM^{(k)}(\delta_1,\delta_2,\ldots,\delta_n)=\left(\begin{array}{l}\sqrt[q]{1-\exp\left\{-\left[\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(-\ln\left(1-\exp\left\{-\left[\frac{1}{k}\sum_{j=1}^{k}w_{i_j}\left(-\ln(\tau_{i_j}^q)\right)^{\varphi}\right]^{1/\varphi}\right\}\right)\right)^{\varphi}\right]^{1/\varphi}\right\}},\\ \sqrt[q]{\exp\left\{-\left[\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(-\ln\left(1-\exp\left\{-\left[\frac{1}{k}\sum_{j=1}^{k}w_{i_j}\left(-\ln(1-\eta_{i_j}^q)\right)^{\varphi}\right]^{1/\varphi}\right\}\right)\right)^{\varphi}\right]^{1/\varphi}\right\}},\\ \sqrt[q]{\exp\left\{-\left[\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(-\ln\left(1-\exp\left\{-\left[\frac{1}{k}\sum_{j=1}^{k}w_{i_j}\left(-\ln(1-\vartheta_{i_j}^q)\right)^{\varphi}\right]^{1/\varphi}\right\}\right)\right)^{\varphi}\right]^{1/\varphi}\right\}}\end{array}\right) \quad (25)$$

$$TSFWDBM^{s,t}(\delta_1,\delta_2,\ldots,\delta_n)=\left(\begin{array}{l}\left(\frac{1}{1+\left(\frac{1}{\frac{s+t}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(\frac{1}{\frac{s}{w_i\left(\frac{\tau_i^q}{1-\tau_i^q}\right)^{\gamma}}+\frac{t}{w_j\left(\frac{\tau_j^q}{1-\tau_j^q}\right)^{\gamma}}}\right)}\right)^{1/\gamma}}\right)^{1/q},\left(1-\frac{1}{1+\left(\frac{1}{\frac{s+t}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(\frac{1}{\frac{s}{w_i\left(\frac{1-\eta_i^q}{\eta_i^q}\right)^{\gamma}}+\frac{t}{w_j\left(\frac{1-\eta_j^q}{\eta_j^q}\right)^{\gamma}}}\right)}\right)^{1/\gamma}}\right),\\ \left(1-\frac{1}{1+\left(\frac{1}{\frac{s+t}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(\frac{1}{\frac{s}{w_i\left(\frac{1-\eta_i^q}{\eta_i^q}\right)^{\gamma}}+\frac{t}{w_j\left(\frac{1-\eta_j^q}{\eta_j^q}\right)^{\gamma}}}\right)}\right)^{1/\gamma}}\right)^{1/q}\end{array}\right) \quad (26)$$

Therefore, the proposed methods are more advantageous than existing methods in capturing attribute associations and eliminating counter-intuition. To further explain the merits of the developed operators, some features are compared with the available aggregation operators. Their features are exhibited in Table 11.

Table 11 Comparison of features of different aggregation operators

AOs	Whether process information is more powerful	Whether any two attributes are correlated	Whether any multiple attributes are correlated	Whether it eliminate counterintuitive	Whether it has decision-making flexibility	Whether it reflects decision-makers' attitude or preferences
q-ROFIWHM[42]	No	Yes	Yes	Yes	Yes	No
PFNWBM[43]	No	Yes	No	No	Yes	No
SFDWHeM[44]	No	Yes	No	No	Yes	Yes
TSFWA [21]	Yes	No	No	No	No	No
TSFWG [21]	Yes	No	No	No	No	No
TSFWAI[24]	Yes	No	No	Yes	No	No
TSFWGI[24]	Yes	No	No	Yes	No	No
TSFAAWHM[29]	Yes	Yes	Yes	No	Yes	Yes
TSFWGMSM[4	Yes	Yes	Yes	No	Yes	No

5]						
TSFWDBM[32]	Yes	Yes	No	No	Yes	Yes
TSFIPWHeM[25]	Yes	Yes	No	Yes	Yes	No
TSFWIPBM[46]	Yes	Yes	No	Yes	Yes	No
TSFAdWHM	Yes	Yes	Yes	Yes	Yes	Yes
TSFAdWDHM	Yes	Yes	Yes	Yes	Yes	Yes

From Table 11, the designed operators have the below merits more current aggregation operators [21,24-25,29,32,42-46].

(1) In terms of evaluation information expression, this paper adopts TSFNs to portray the subjective judgment and evaluation degree of different experts when dealing with the MAGDM problem. The TSFS has a ternary mathematical structure to express the personalized preference information. It enables the flexibility to realize the freedom of expression and adjust the decision space of the judge, which are not available in q-rung orthopair fuzzy set, picture fuzzy set and spherical fuzzy set. Therefore, the proposed aggregation operators have stronger information processing capability than the existing q-ROFIWHM [42], PFNWBM [43] and SFDWHeM [44] operators.

(2) In terms of attribute correlation, the TSFWA and TSFWG [21], TSFWAI and TSFWGI [24] operators are unable to capture the interrelationship between attributes. The TSFWDBM [32], TSFIPWHeM [25] and TSFWIPBM [46] operators contain HeM and BM, but these aggregation operators can only capture the interrelationship between two attributes and their computational complexity is higher than the developed aggregation operators. In contrast, the developed aggregation operators not only have the ability to capture multiple attribute correlations, but also have the flexibility to determine the number of associated attributes based on realistic decision-making situations.

(3) In terms of eliminating counter-intuition, although the TSFAAWHM [29] and TSFWGMSM [45] operators can capture the interrelationship among multiple attributes, they cannot eliminate counter-intuition in the information fusion process. The TSFWAI [24], TSFWGI [24], TSFIPWHeM [25] and TSFWIPBM [46] operators are all based on IOLs, which enable interactive computation between MD, AD and ND during the aggregation of TSFNs. The IOLs can realize the interaction between the values of each element to avoid the situation where each element is zero-valued. Compared with the IOLs, however, the AdOLs of TSFNs not only have lower computational complexity than IOLs, but also can realize decision flexibility and reflect the decision preferences of decision-makers by adjusting the parameters according to the real decision-making needs.

(4) In terms of decision flexibility, none of the TSFWA and TSFWG [21], TSFWAI and TSFWGI [24] operators contain adjustable computational parameters, and thus none of them can achieve decision flexibility. The TSFWIPBM [46], TSFIPWHeM [25] and TSFWGMSM [45] operators contain some association parameters which can reflect the number of associated attributes and their degree of association. They can show the flexibility of decision-making, but they cannot reflect the decision-makers' decision preferences. The TSFAAWHM [29] and TSFWDBM [32] operators were developed based on AAOLs and DOLs containing computational parameters, respectively. Like the proposed AdOLs, these parameters enable decision-making flexibility while reflecting the decision-makers' preferences. Unfortunately, the TSFAAWHM [29] and TSFWDBM [32] operators fail to obtain results when dealing with TSFNs where any of the elements is zero-valued. Therefore, the

advanced aggregation operators are more versatile and practical in tackling realistic MAGDM challenges.

Summarizing the above quantitative and qualitative comparisons, the AdOLs and HM/DHM are integrated and developed to form a novel TSF aggregation operator in the TSF environment, while we provide a flexible and generalized decision-making models based on the TSFAdWHM and TSFAdWDHM operators to solve complex TSF MAGDM issues, which feature a powerful capability of information representation, eliminating counter-intuition, and capturing the interrelationship among multiple attributes. As a result, the developed methods are more powerful and advantageous than existing models.

6 Conclusions

To fuse TSF information more efficiently, we defined the AdOLs for TSFNs and integrate them with HM to develop the TSFAdWHM and its dual shape. Further, two new TSF MAGDM models were constructed based on the raised operators. Lastly, we provided some examples to explain the feasibility, effectiveness, and superiorities of the presented methods. The results of the sensitivity analysis and comparative study showed that the developed methods are not only capable of capturing the interrelationship among multiple attributes, but also eliminating counter-intuitive as well as reflecting the decision-makers' preferences. The proposed methods outperform the existing TSF aggregation operator-based methods for solving complex MAGDM problems.

However, the proposed methods still have three drawbacks. (1) The importance of experts and attributes are given in this paper without considering the actual decision-making needs. For example, the expert and attribute significance is calculated based on the characteristics of inter-expert relationship network and subjective and objective combination evaluation of attributes, etc. (2) In the developed TSFAdWHM and TSFAdWDHM operators, the number of associated attributes is recognized as fixed during the application of this paper, without taking into account the diversified number of associated attributes in the actual decision-making issues. (3) The aggregation operator-based models developed in this article ignore the priority order between different attributes, but this is objectively present in the priority order of attributes in many real situations.

The AdOLs will be extended to other decision-making environments in the future, such as neutrosophic sets [47], uncertain linguistic sets [48], probabilistic linguistic sets [49], etc., to develop different types of aggregation operators integrating some innovative features such as prioritized, partitioned and induced information. In addition, the AdOLs will be integrated with alternatives ranking techniques such as ARAS [50], WASPAS [51], CoCoSo [52], and MULTIMOORA [53] for applying and solving complex real-life MAGDM challenges, for instance, business investment decisions, technology evaluation and selection, and location selection.

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