Journal of Intelligent Decision Making
and Information Science

Journal homepage: www.jidmis.org
eISSN: 3079-0875

Construction of New Similarity Measures for Complex Pythagorean Fuzzy Sets and Their Applications in Decision-Making Problems

Donglai Wang[1], Sukumar Letchmunan[2], Juan Liao[2], Haoye Qiu[3], Zhe Liu[2,4,*]

[1] College of Computer Science, Chongqing University, Chongqing 400044, China
[2] School of Computer Sciences, Universiti Sains Malaysia, Penang 11800, Malaysia
[3] School of Computer Science and Engineering, Southeast University, Nanjing 211189, China
[4] College of Mathematics and Computer, Xinyu University, Xinyu 338004, China

ARTICLE INFO

Article history:
Received 01 December 2024
Received in revised form 30 December 2024
Accepted 02 January 2025
Available online 03 January 2025

Keywords:
Complex Pythagorean fuzzy sets; similarity measures; trigonometric function; decision-making

ABSTRACT

Complex Pythagorean fuzzy sets (CPFSs) extend Pythagorean fuzzy sets by expressing membership, non-membership, and hesitancy degrees using complex numbers. An urgent issue in CPFSs is determining how to accurately measure the similarity between sets. Similarity measures are crucial for assessing the closeness between two objects and are extensively applied in decision-making. In this paper, we propose new similarity measures based on trigonometric functions and their weighted representations. Additionally, we investigate the properties these measures satisfy and demonstrate their effectiveness through several numerical examples. Lastly, we apply these similarity measures to decision-making problems, including pattern recognition and medical diagnosis.

1. Introduction

Similarity measure is an important concept in probability theory, as it serves to evaluate the degree of resemblance between two variables or samples. The similarity between two objects represents a numerical indication of how much they resemble each other. As a result, objects that are more alike exhibit higher levels of similarity. The probability theory-based similarity measure method excels in handling specific information scenarios. Nevertheless, in practical scenarios, decision-makers often encounter uncertain information, rendering these methods inadequate and unable to yield precise outcomes when confronted with such uncertainties.

*Corresponding author.
E-mail address: liuzhe921@gmail.com

https://doi.org/10.59543/jidmis.v2i.11737

Up to now, various theories have been developed, including fuzzy sets [24, 27, 35], evidence theory [20, 22, 23], and neutrosophic sets [25, 26, 30]. The introduction of fuzzy set theory by [40] marked a significant milestone in this field, forming the basis for fuzzy decision-making. Building on this, Atanassov [7] proposed intuitionistic fuzzy sets (IFSs), which include both membership and non-membership degrees, restricting their combined total to 1 or less. Since then, IFSs have been applied across a range of decision-making areas [11, 14, 29, 41]. However, this constraint limits their effectiveness in practical problem-solving. To address the limitation, Yager [37] introduced Pythagorean fuzzy sets (PFSs), which ensure that the sum of the squares of the membership and non-membership degrees is no greater than 1. Due to their advantages, PFSs have attracted significant research interest and have been applied in various decision-making contexts [6, 13, 16, 21]. Despite these advancements, PFSs still offer opportunities for further exploration and refinement.

Over the years, numerous applications of traditional FSs theory models have been explored. However, due to their reliance on real-valued membership grades, these models were unable to effectively represent two-dimensional vague data. A significant advancement in FSs theory came with the introduction of complex fuzzy sets (CFSs) in [32], which expanded the framework to include complex-valued membership, allowing for the representation of phase information and multidimensional attributes. Building on this, the literature [4] proposed complex intuitionistic fuzzy sets (CIFSs), incorporating two complex functions that represent membership and non-membership degrees. Following this development, extensive research has been conducted on CIFSs [3, 10, 19, 28, 33].

However, CIFSs face similar limitation to IFSs, leading [34] to extend CIFSs into complex Pythagorean fuzzy sets (CPFSs), which have since attracted considerable attention[1, 8, 15]. For instance, the literature [31] introduced new complex Pythagorean fuzzy Einstein weighted geometric and hybrid geometric operators, aimed at mitigating the transmission rate of COVID-19. Furthermore, Wu et al. [36] highlighted that the distance measure proposed in [34] does not satisfy the axiomatic criteria for CPFSs, prompting the development of novel distance measures based on CPFSs. Liu et al. [18] designed a new class of Archimedean aggregation operators tailored for CPFSs to improve decision-making processes. Hezam et al. [12] introduced complex Pythagorean fuzzy geometric aggregation operators to address multicriteria group decision-making problems. Additionally, Liu et al. [17] developed Dombi aggregation operators based on CPFSs information, applying them to solve green supply chain management issues within a complex Pythagorean fuzzy context.

Despite many research efforts and notable progress within CPFSs, there remains a gap in the development of similarity measures for CPFSs. Some existing similarity measures may lead to counterintuitive results for various reasons. Trigonometric functions, which are fundamental mathematical tools, have been widely applied in the study of similarity measures for IFSs and PFSs [5, 38, 39]. Recently, Ali [2] introduced trigonometric function-based similarity measures for CFSs. To capture more uncertain information and expand the range of applications, we propose a set of similarity measures for CPFSs based on trigonometric functions, along with their weighted forms. Owing to the inherent characteristics of the trigonometric functions, our proposed measures effectively identify the difference between CPFSs.

The main contributions are summarized below:

1. We introduce a novel set of similarity measures based on trigonometric functions, specifically incorporating sine, cosine, tangent and cotangent functions

2. We show that the proposed measure satisfies the necessary properties and provide numerical examples to support the validity of the measures.

3. We apply these similarity measures to various decision-making problems to showcase their practical utility.

This article is organized as follows: In section 2, we briefly introduce the basic concepts of CFS,CIFS,CPFS. We introduce some existing similarity measures and propose the trigonometric similarity measures and weighted similarity measures in section 3. Then, in section 4, we verify the properties that our presented similarity measures hold through some numerical examples. In section 5, we apply these measures to pattern recognition and medical diagnosis. Section 6 concludes the papers.

2. Preliminary

This section will provide some basic concept about CFS, CIFS and CPFS.

2.1 Complex fuzzy set

Definition 1 *[32] Assume that Φ is a finite universe of discourse (UOD). The complex fuzzy set (CFS) $\mathcal{C}$ in Φ is defined below:*

$$\mathcal{C} = \{\langle \mathcal{M}_{\mathcal{C}}(\phi)\rangle | \phi \in \Phi\} \tag{1}$$

where $\mathcal{M}_{\mathcal{C}} : \Phi \to \{y : y \in C, |y| \leq 1\}$ is the complex-valued membership degree, which is denoted as $\mathcal{M}_{\mathcal{C}} = \mathscr{X}_{\mathcal{C}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathcal{C}}(\phi)}}$ where $0 \leq \mathscr{X}_{\mathcal{C}(\phi)} \leq 1$ and $0 \leq \mathscr{W}_{\mathcal{C}}(\phi) \leq 1$.

2.2 Complex intuitionistic fuzzy set

Definition 2 *[4] Assume that Φ is a finite UOD. The complex intuitionistic fuzzy set (CIFS) $\mathcal{I}$ in Φ is defined below:*

$$\mathcal{I} = \{\langle \mathcal{M}_{\mathcal{I}}(\phi), \mathcal{N}_{\mathcal{I}}(\phi)\rangle | \phi \in \Phi\} \tag{2}$$

where $\mathcal{M}_{\mathcal{I}}, \mathcal{N}_{\mathcal{I}} : \Phi \to \{y : y \in C, |y| \leq 1\}$ are the complex-valued membership and non-membership degrees, which are denoted as $\mathcal{M}_{\mathcal{I}} = \mathscr{X}_{\mathcal{I}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathcal{I}}(\phi)}}, \mathcal{N}_{\mathcal{I}} = \mathscr{Y}_{\mathcal{I}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathcal{I}}(\phi)}}$, where $\mathscr{X}_{\mathcal{I}(\phi)}$, $\mathscr{Y}_{\mathcal{I}(\phi)} \in [0,1]$ and $\mathscr{X}_{\mathcal{I}(\phi)} + \mathscr{Y}_{\mathcal{I}(\phi)} \leq 1$. Additionally, $\mathscr{W}_{\mathscr{X}_{\mathcal{I}}(\phi)}, \mathscr{W}_{\mathscr{Y}_{\mathcal{I}}(\phi)} \in [0,1]$ and $\mathscr{W}_{\mathscr{X}_{\mathcal{I}}(\phi)} + \mathscr{W}_{\mathscr{Y}_{\mathcal{I}}(\phi)} \leq 1$. Moreover, the hesitancy degree is defined as $\mathcal{H}_{\mathcal{I}}(\phi) = \mathscr{H}_{\mathcal{I}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{H}_{\mathcal{I}}(\phi)}}$, where $\mathscr{H}_{\mathcal{I}}(\phi) = 1 - \mathscr{X}_{\mathcal{I}}(\phi) - \mathscr{Y}_{\mathcal{I}}(\phi)$ and $\mathscr{W}_{\mathscr{H}_{\mathcal{I}}(\phi)} = 1 - \mathscr{W}_{\mathscr{X}_{\mathcal{I}}(\phi)} - \mathscr{W}_{\mathscr{Y}_{\mathcal{I}}(\phi)}$.

2.3 Complex pythagorean fuzzy set

Definition 3 *[34] Assume that Φ is a finite UOD. The complex Pythagorean fuzzy set (CPFS) $\mathbb{P}$ in Φ is defined below:*

$$\mathbb{P} = \{\langle \mathcal{M}_{\mathbb{P}}(\phi), \mathcal{N}_{\mathbb{P}}(\phi)\rangle | \phi \in \Phi\} \tag{3}$$

where $\mathcal{M}_{\mathbb{P}}, \mathcal{N}_{\mathbb{P}} : \Phi \to \{y : y \in C, |y| \leq 1\}$ are the complex-valued membership and non-membership degrees, which are denoted as $\mathcal{M}_{\mathbb{P}} = \mathscr{X}_{\mathbb{P}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathbb{P}}(\phi)}}, \mathcal{N}_{\mathbb{P}} = \mathscr{Y}_{\mathbb{P}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathbb{P}}(\phi)}}$, where $\mathscr{X}_{\mathbb{P}(\phi)}$, $\mathscr{Y}_{\mathbb{P}(\phi)} \in [0,1]$ and $\mathscr{X}^2_{\mathbb{P}(\phi)} + \mathscr{Y}^2_{\mathbb{P}(\phi)} \leq 1$. Additionally, $\mathscr{W}_{\mathscr{X}_{\mathbb{P}}(\phi)}, \mathscr{W}_{\mathscr{Y}_{\mathbb{P}}(\phi)} \in [0,1]$ and $\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}}(\phi)} + \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}}(\phi)} \leq 1$. Moreover, the hesitancy degree is defined as $\mathcal{H}_{\mathbb{P}}(\phi) = \mathscr{H}_{\mathbb{P}}(\phi) \cdot e^{2\pi i \mathscr{W}_{\mathscr{H}_{\mathbb{P}}(\phi)}}$, where $\mathscr{H}_{\mathbb{P}}(\phi) = \sqrt{1 - \mathscr{X}^2_{\mathbb{P}}(\phi) - \mathscr{Y}^2_{\mathbb{P}}(\phi)}$ and $\mathscr{W}_{\mathscr{H}_{\mathbb{P}}(\phi)} = \sqrt{1 - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}}(\phi)} - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}}(\phi)}}$.

3. Similarity Measures for CPFSs

In this section, we first review some existing similarity measures, then we define some novel similarity measures between two CPFSs $\mathbb{P}_1 = \{\langle \phi_i, \mathscr{X}_{\mathbb{P}_1}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}}, \mathscr{Y}_{\mathbb{P}_1}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}(\phi_i)}}\rangle | \phi_i \in \Phi\}$ and $\mathbb{P}_2 = \{\langle \phi_i, \mathscr{X}_{\mathbb{P}_2}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}}, \mathscr{Y}_{\mathbb{P}_2}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}(\phi_i)}}\rangle | \phi_i \in \Phi\}$ on UOD Φ.

3.1 Existing Similarity Measures

Definition 4 *Similarity measures based on the distance measures for CPFSs proposed by Wu et al. [36].*

$$S_{Wu}(\mathbb{P}_1,\mathbb{P}_2) = 1-\frac{1}{2m}\sum_{i=1}^{m}\begin{bmatrix} \frac{1}{4}\left(|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\right) \\ +\frac{1}{2}\max\left(|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\right) \\ +\frac{1}{4}\left(|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\right) \\ +\frac{1}{2}\max\left(|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)|, |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)|, |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\right) \end{bmatrix} \tag{4}$$

$$S_{SK}(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{4m}\sum_{i=1}^{m}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix} \tag{5}$$

$$S_{WX}(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{4m}\sum_{i=1}^{m}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix} \tag{6}$$

$$S_{G}(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{2m}\sum_{i=1}^{m}\begin{pmatrix} \max\{|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\} \\ +\max\{|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)|, |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)|\} \end{pmatrix} \tag{7}$$

$$S_{YC}(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{2m}\sum_{i=1}^{m}\begin{pmatrix} \max\{|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\} \\ +\max\{|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)|, |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)|, |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\} \end{pmatrix} \tag{8}$$

$$S^{(1)}_{Wu} = 1 - \frac{1}{2m}\sum_{i=1}^{m}\begin{pmatrix} \sqrt{\frac{1}{2}\left(L\left(1-\mathscr{X}^2_{\mathbb{P}_1}(\phi_i), 1-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)\right)\right) + L\left(\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i), \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)\right)} \\ +\sqrt{\frac{1}{2}\left(L\left(1-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i), 1-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)\right)\right) + L\left(\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i), \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i)\right)} \end{pmatrix} \tag{9}$$

where $L(\alpha,\beta) = \alpha\log_2\frac{2\alpha}{\alpha+\beta} + \beta\log_2\frac{2\beta}{\alpha+\beta}$.

3.2 Proposed Similarity Measures for CPFSs

In this section, we will introduce some similarity measures based on trigonometric functions.

Definition 5 *For two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, the similarity measures (S^1_{sin}, S^2_{sin}) based on sine function are defined as:*

$$S_{sin}^1(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_2}^2(\phi_i)| + |\mathscr{Y}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_2}^2(\phi_i)| \\ +|\mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i)| + |\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i)| \\ +|\mathscr{H}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_2}^2(\phi_i)| + |\mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i)| \end{pmatrix}\right) \tag{10}$$

$$S_{sin}^2(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{2}\begin{pmatrix} |\mathscr{X}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_2}^2(\phi_i)| \vee |\mathscr{Y}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_2}^2(\phi_i)| \\ \vee|\mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i)| \vee |\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i)| \\ \vee|\mathscr{H}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_2}^2(\phi_i)| \vee |\mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i)| \end{pmatrix}\right) \tag{11}$$

Theorem 1 *Considering three CPFSs $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$, $S_{sin}^k(k=1,2)$ holds the following properties:*

1. $S_{sin}^k(\mathbb{P}_1,\mathbb{P}_2) = S_{sin}^k(\mathbb{P}_2,\mathbb{P}_1)$
2. $S_{sin}^k(\mathbb{P}_1,\mathbb{P}_2) = 1$ *iff* $\mathbb{P}_1 = \mathbb{P}_2$
3. $0 \leq S_{sin}^k(\mathbb{P}_1,\mathbb{P}_2) \leq 1$
4. *If* $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$, *then* $S_{sin}^k(\mathbb{P}_1,\mathbb{P}_3) \leq S_{sin}^k(\mathbb{P}_1,\mathbb{P}_2)$ *and* $S_{sin}^k(\mathbb{P}_1,\mathbb{P}_3) \leq S_{sin}^k(\mathbb{P}_2,\mathbb{P}_3)$

Proof 1 S_{sin}^1 *as an example, for two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, we have:*

$$S_{sin}^1(\mathbb{P}_1,\mathbb{P}_2) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_2}^2(\phi_i)| + |\mathscr{Y}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_2}^2(\phi_i)| \\ +|\mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i)| + |\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i)| \\ +|\mathscr{H}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_2}^2(\phi_i)| + |\mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i)| \end{pmatrix}\right)$$

$$S_{sin}^1(\mathbb{P}_2,\mathbb{P}_1) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_1}^2(\phi_i)| + |\mathscr{Y}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_1}^2(\phi_i)| \\ +|\mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i)| + |\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i)| \\ +|\mathscr{H}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_1}^2(\phi_i)| + |\mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i)| \end{pmatrix}\right)$$

Obviously,

$$\begin{aligned} &|\mathscr{X}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_2}^2(\phi_i)| = |\mathscr{X}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{X}_{\mathbb{P}_1}^2(\phi_i)| \\ &|\mathscr{Y}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_2}^2(\phi_i)| = |\mathscr{Y}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{Y}_{\mathbb{P}_1}^2(\phi_i)| \\ &|\mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i)| = |\mathscr{W}_{\mathscr{X}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{X}_{\mathbb{P}_1}}^2(\phi_i)| \\ &|\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i)| = |\mathscr{W}_{\mathscr{Y}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{Y}_{\mathbb{P}_1}}^2(\phi_i)| \\ &|\mathscr{H}_{\mathbb{P}_1}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_2}^2(\phi_i)| = |\mathscr{H}_{\mathbb{P}_2}^2(\phi_i) - \mathscr{H}_{\mathbb{P}_1}^2(\phi_i)| \\ &|\mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i)| = |\mathscr{W}_{\mathscr{H}_{\mathbb{P}_2}}^2(\phi_i) - \mathscr{W}_{\mathscr{H}_{\mathbb{P}_1}}^2(\phi_i)| \end{aligned}$$

Thus, we can obtain $S_{sin}^1(\mathbb{P}_1,\mathbb{P}_2)$ *=*$S_{sin}^1(\mathbb{P}_2,\mathbb{P}_1)$

Proof 2 S^1_{sin} *as an example, for two CPFSs* $\mathbb{P}_1$ *and* $\mathbb{P}_2$*, assume* $\mathbb{P}_1$ = $\mathbb{P}_2$*, then we have:*

$$\begin{aligned}
&\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{X}^2_{\mathbb{P}_2}(\phi_i), \mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i), \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i),\\
&\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i), \mathscr{H}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{H}^2_{\mathbb{P}_2}(\phi_i), \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)\\
\Rightarrow &|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| = 0, |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| = 0,\\
&|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| = 0, |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| = 0\\
&|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| = 0, |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| = 0\\
\Rightarrow &\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix} = 0\\
\Rightarrow &\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) = 0
\end{aligned}$$

Hence, we can obtain:

$$S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) = 1$$

Considering $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) = 1$*, for any* $\phi_i \in \Phi$*, we have:*

$$S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) = 1$$

Therefore, we can obtain:

$$\begin{aligned}
&|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| = 0, |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| = 0\\
&|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| = 0, |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| = 0\\
&|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| = 0, |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| = 0\\
\Rightarrow &\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{X}^2_{\mathbb{P}_2}(\phi_i), \mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i), \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i),\\
&\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i), |\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) = \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|, |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) = \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|
\end{aligned}$$

Thus, we can prove that $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) = 1$ *iff* $\mathbb{P}_1 = \mathbb{P}_2$

Proof 3 *S^1_{sin} as an example, for two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, we can obtain:*

$$
\begin{aligned}
&0 \leq \mathscr{X}^2_{\mathbb{P}_1}(\phi_i), \mathscr{X}^2_{\mathbb{P}_2}(\phi_i), \mathscr{Y}^2_{\mathbb{P}_1}(\phi_i), \mathscr{Y}^2_{\mathbb{P}_1}(\phi_i), \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i), \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i), \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i), \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i) \leq 1 \\
&\Rightarrow 0 \leq \begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix} \leq 1 \\
&\Rightarrow 0 \leq \sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) \leq 1 \\
&\Rightarrow 0 \leq 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) \leq 1
\end{aligned}
$$

Therefore, we can prove $0 \leq S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) \leq 1$.

Proof 4 *S^1_{sin} as an example, for three CPFSs $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$, considering $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$, then we have:*

$$
\begin{aligned}
&\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) \leq \mathscr{X}^2_{\mathbb{P}_2}(\phi_i) \leq \mathscr{X}^2_{\mathbb{P}_3}(\phi_i), \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) \leq \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i) \leq \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_3}}(\phi_i) \\
&\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) \geq \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i) \geq \mathscr{Y}^2_{\mathbb{P}_3}(\phi_i), \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) \geq \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i) \geq \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_3}}(\phi_i)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| \leq |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_3}(\phi_i)| \\
&|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| \leq |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_3}}(\phi_i)| \\
&|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \leq |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_3}(\phi_i)| \\
&|\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \leq |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_3}}(\phi_i)|
\end{aligned}
$$

Hence, we can obtain:

$$
\begin{aligned}
S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) &= 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) \\
&\leq 1 - \frac{1}{m}\sum_{i=1}^{m}\sin\left(\frac{\pi}{8}\begin{pmatrix} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}}(\phi_i)| + |\mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{Y}_{\mathbb{P}_2}}(\phi_i)| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{pmatrix}\right) \\
&= S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2)
\end{aligned}
$$

We can get $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^1_{sin}(\mathbb{P}_2, \mathbb{P}_3)$ in the same way. Therefore,we can prove that if $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$, then $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2)$ and $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^1_{sin}(\mathbb{P}_2, \mathbb{P}_3)$.

Definition 6 *For two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, the similarity measures (S^1_{cos}, S^2_{cos}) between CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$ based on cosine function are defined as:*

$$S_{cos}^1(\mathbb{P}_1,\mathbb{P}_2)=\frac{1}{m}\sum_{i=1}^{m}\cos\left(\frac{\pi}{8}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\+|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(12)$$

$$S_{cos}^2(\mathbb{P}_1,\mathbb{P}_2)=\frac{1}{m}\sum_{i=1}^{m}\cos\left(\frac{\pi}{2}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\\vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(13)$$

Theorem 2 *Considering three CPFSs $\mathbb{P}_1,\mathbb{P}_2$ and $\mathbb{P}_3$, $S_{cos}^k(k=1,2)$ holds the following properties:*

1. $S_{cos}^k(\mathbb{P}_1,\mathbb{P}_2)=S_{cos}^k(\mathbb{P}_2,\mathbb{P}_1)$
2. $S_{cos}^k(\mathbb{P}_1,\mathbb{P}_2)=1$ *iff* $\mathbb{P}_1=\mathbb{P}_2$
3. $0\le S_{cos}^k(\mathbb{P}_1,\mathbb{P}_2)\le 1$
4. *If* $\mathbb{P}_1\subseteq\mathbb{P}_2\subseteq\mathbb{P}_3$, *then* $S_{cos}^k(\mathbb{P}_1,\mathbb{P}_3)\le S_{cos}^k(\mathbb{P}_1,\mathbb{P}_2)$ *and* $S_{cos}^k(\mathbb{P}_1,\mathbb{P}_3)\le S_{cos}^k(\mathbb{P}_2,\mathbb{P}_3)$

Definition 7 *For two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, the similarity measures (S_{tan}^1, S_{tan}^2) between CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$ based on tangent function are defined as:*

$$S_{tan}^1(\mathbb{P}_1,\mathbb{P}_2)=1-\frac{1}{m}\sum_{i=1}^{m}\tan\left(\frac{\pi}{16}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\+|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(14)$$

$$S_{tan}^2(\mathbb{P}_1,\mathbb{P}_2)=1-\frac{1}{m}\sum_{i=1}^{m}\tan\left(\frac{\pi}{4}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\\vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(15)$$

Theorem 3 *Considering three CPFSs $\mathbb{P}_1,\mathbb{P}_2$ and $\mathbb{P}_3$, $S_{tan}^k(k=1,2)$ holds the following properties:*

1. $S_{tan}^k(\mathbb{P}_1,\mathbb{P}_2)=S_{tan}^k(\mathbb{P}_2,\mathbb{P}_1)$
2. $S_{tan}^k(\mathbb{P}_1,\mathbb{P}_2)=1$ *iff* $\mathbb{P}_1=\mathbb{P}_2$
3. $0\le S_{tan}^k(\mathbb{P}_1,\mathbb{P}_2)\le 1$
4. *If* $\mathbb{P}_1\subseteq\mathbb{P}_2\subseteq\mathbb{P}_3$, *then* $S_{tan}^k(\mathbb{P}_1,\mathbb{P}_3)\le S_{tan}^k(\mathbb{P}_1,\mathbb{P}_2)$ *and* $S_{tan}^k(\mathbb{P}_1,\mathbb{P}_3)\le S_{tan}^k(\mathbb{P}_2,\mathbb{P}_3)$

Definition 8 *For two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, the similarity measures (S_{cot}^1, S_{cot}^2) between CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$ based on cotangent function are defined as:*

$$S_{cot}^1(\mathbb{P}_1,\mathbb{P}_2)=\frac{1}{m}\sum_{i=1}^{m}\cot\left(\frac{\pi}{4}+\frac{\pi}{16}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|+|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\+|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|+|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(16)$$

$$S_{cot}^2(\mathbb{P}_1,\mathbb{P}_2)=\frac{1}{m}\sum_{i=1}^{m}\cot\left(\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}|\mathscr{X}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{X}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)|\\\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)}-\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}|\\\vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i)-\mathscr{H}^2_{\mathbb{P}_2}(\phi_i)|\vee|\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i)-\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)|\end{pmatrix}\right)\quad(17)$$

Theorem 4 *Considering three CPFSs $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$, $S^k_{cot}(k = 1, 2)$ holds the following properties:*

1. $S^k_{cot}(\mathbb{P}_1, \mathbb{P}_2) = S^k_{cot}(\mathbb{P}_2, \mathbb{P}_1)$
2. $S^k_{cot}(\mathbb{P}_1, \mathbb{P}_2) = 1$ *iff* $\mathbb{P}_1 = \mathbb{P}_2$
3. $0 \leq S^k_{cot}(\mathbb{P}_1, \mathbb{P}_2) \leq 1$
4. *If* $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$, *then* $S^k_{cot}(\mathbb{P}_1, \mathbb{P}_3) \leq S^k_{cot}(\mathbb{P}_1, \mathbb{P}_2)$ *and* $S^k_{cot}(\mathbb{P}_1, \mathbb{P}_3) \leq S^k_{cot}(\mathbb{P}_2, \mathbb{P}_3)$

3.3 *Proposed Weighted Similarity Measures for CPFSs*

In this section, we will further introduce the weighted similarity measures based on sine, cosine, tangent and cotangent functions.

Definition 9 *For two CPFSs $\mathbb{P}_1$ and $\mathbb{P}_2$, the weighted similarity measures based on sine, cosine, tangent and cotangent functions are defined as:*

$$S^1_{wsin}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \sum_{i=1}^{m} \omega_i \sin\left(\frac{\pi}{8}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| + |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{18}$$

$$S^2_{wsin}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \sum_{i=1}^{m} \omega_i \sin\left(\frac{\pi}{2}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ \vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \vee |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ \vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{19}$$

$$S^1_{wcos}(\mathbb{P}_1, \mathbb{P}_2) = \sum_{i=1}^{m} \omega_i \cos\left(\frac{\pi}{8}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| + |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{20}$$

$$S^2_{wcos}(\mathbb{P}_1, \mathbb{P}_2) = \sum_{i=1}^{m} \omega_i \cos\left(\frac{\pi}{2}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ \vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \vee |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ \vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{21}$$

$$S^1_{wtan}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \sum_{i=1}^{m} \omega_i \tan\left(\frac{\pi}{16}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| + |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{22}$$

$$S^2_{wtan}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \sum_{i=1}^{m} \omega_i \tan\left(\frac{\pi}{4}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ \vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \vee |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ \vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{23}$$

$$S^1_{wcot}(\mathbb{P}_1, \mathbb{P}_2) = \sum_{i=1}^{m} \omega_i \cot\left(\frac{\pi}{4} + \frac{\pi}{16}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ +|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| + |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ +|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| + |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{24}$$

$$S^2_{wcot}(\mathbb{P}_1, \mathbb{P}_2) = \sum_{i=1}^{m} \omega_i \cot\left(\frac{\pi}{4} + \frac{\pi}{4}\left(\begin{array}{r} |\mathscr{X}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{X}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{Y}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{Y}^2_{\mathbb{P}_2}(\phi_i)| \\ \vee|\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \vee |\mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_1}(\phi_i)} - \mathscr{W}^2_{\mathscr{X}_{\mathbb{P}_2}(\phi_i)}| \\ \vee|\mathscr{H}^2_{\mathbb{P}_1}(\phi_i) - \mathscr{H}^2_{\mathbb{P}_2}(\phi_i)| \vee |\mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_1}}(\phi_i) - \mathscr{W}^2_{\mathscr{H}_{\mathbb{P}_2}}(\phi_i)| \end{array}\right)\right) \tag{25}$$

Theorem 5 *Considering three CPFSs $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$, the weighted similarity measures hold following properties(e.g.S^k_{wsin}):*

1. $0 \leq S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_2) \leq 1$
2. $S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_2) = 1$ *iff* $\mathbb{P}_1 = \mathbb{P}_2$
3. $S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_2) = S^k_{wsin}(\mathbb{P}_2, \mathbb{P}_1)$
4. *If* $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$, *then* $S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_2)$ *and* $S^k_{wsin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^k_{wsin}(\mathbb{P}_2, \mathbb{P}_3)$

Proof 5 *The proofs are similar to Theorem 1.*

4. Numerical Examples

Example 1 *There are two CPFSs $\mathbb{P}_1$, $\mathbb{P}_2$ defined on UOD Φ, denoted as follows:*

$$\mathbb{P}_1 = (\mu e^{2\pi i(\nu)}, \nu e^{2\pi i(\mu)}), \mathbb{P}_2 = (\nu e^{2\pi i(\mu)}, \mu e^{2\pi i(\nu)})$$

where μ and ν range from 0 to 1 and satisfy the condition $\mu^2 + \nu^2 \leq 1$, as we can see from Fig 1c,Fig 1d, Fig 1g,Fig 1h.

From Fig 1a,Fig 1b,Fig 1e,Fig 1f, we find that the similarity measures $\mathcal{S}^1_{sin}$, $\mathcal{S}^1_{cos}$, $\mathcal{S}^1_{tan}$, $\mathcal{S}^1_{cot}$ always lie in the range [0,1], even though the parameter μ and nu are changing. What is more, when $\mu = \nu$, the similarity measures $\mathcal{S}^1_{sin}$, $\mathcal{S}^1_{cos}$, $\mathcal{S}^1_{tan}$, $\mathcal{S}^1_{cot}$ obtain the maximum value of 1. Additionally, when $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1$, the similarity measures have the minimum value of 0. Therefore, the property 1 and property 2 are proved. The symmetry property is evidently satisfied by the presented similarity measures, as illustrated in Fig. 1.

Example 2 *There are three CPFSs, expressed as $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ in UOD Φ, which satisfy $\mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \mathbb{P}_3$.*

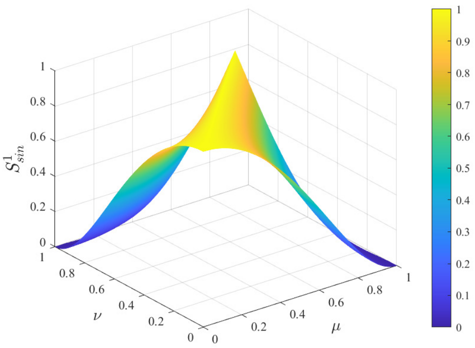
$$\mathbb{P}_1 = (0.2e^{2\pi i(0.25)}, 0.9e^{2\pi i(0.75)})$$
$$\mathbb{P}_2 = (0.5e^{2\pi i(0.35)}, 0.8e^{2\pi i(0.55)})$$
$$\mathbb{P}_3 = (0.6e^{2\pi i(0.45)}, 0.7e^{2\pi i(0.35)})$$

Take S^1_{sin} as an example and we can calculate the results listed below:

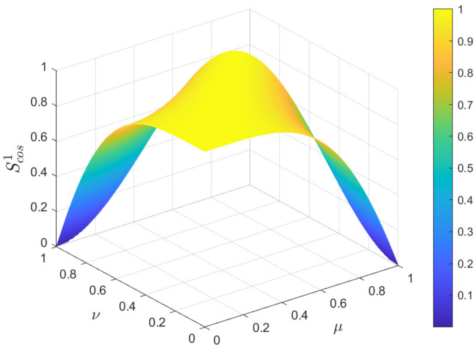
$$S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2) = 1 - \sin\left(\frac{\pi}{8}\left(\begin{array}{c} |0.2^2 - 0.5^2| + |0.9^2 - 0.8^2| + |0.25^2 - 0.35^2| + |0.75^2 - 0.55^2| \\ +|\left(\sqrt{1 - 0.2^2 - 0.9^2}\right)^2 - \left(\sqrt{1 - 0.5^2 - 0.8^2}\right)^2| \\ +|\left(\sqrt{1 - 0.25^2 - 0.75^2}\right)^2 - \left(\sqrt{1 - 0.35^2 - 0.55^2}\right)^2| \end{array}\right)\right) = 0.6392$$

$$S^1_{sin}(\mathbb{P}_2, \mathbb{P}_3) = 1 - \sin\left(\frac{\pi}{8}\left(\begin{array}{c} |0.5^2 - 0.6^2| + |0.8^2 - 0.7^2| + |0.35^2 - 0.45^2| + |0.55^2 - 0.35^2| \\ +\left(|\sqrt{1 - 0.5^2 - 0.8^2}\right)^2 - \left(\sqrt{1 - 0.6^2 - 0.7^2}\right)^2| \\ +|\left(\sqrt{1 - 0.35^2 - 0.55^2}\right)^2 - \left(\sqrt{1 - 0.45^2 - 0.35^2}\right)^2| \end{array}\right)\right) = 0.7437$$

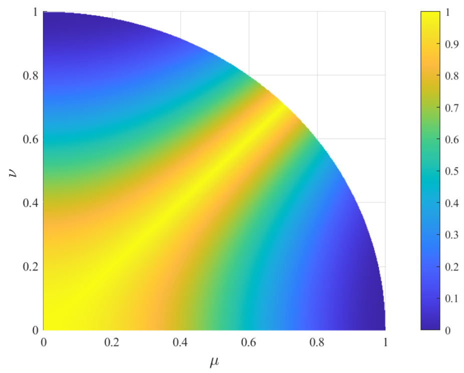
$$S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) = 1 - \sin\left(\frac{\pi}{8}\left(\begin{array}{c} |0.2^2 - 0.6^2| + |0.9^2 - 0.7^2| + |0.25^2 - 0.45^2| + |0.75^2 - 0.35^2| \\ +|\left(\sqrt{1 - 0.2^2 - 0.9^2}\right)^2 - \left(\sqrt{1 - 0.6^2 - 0.7^2}\right)^2| \\ +|\left(\sqrt{1 - 0.25^2 - 0.75^2}\right)^2 - \left(\sqrt{1 - 0.45^2 - 0.35^2}\right)^2| \end{array}\right)\right) = 0.4379$$



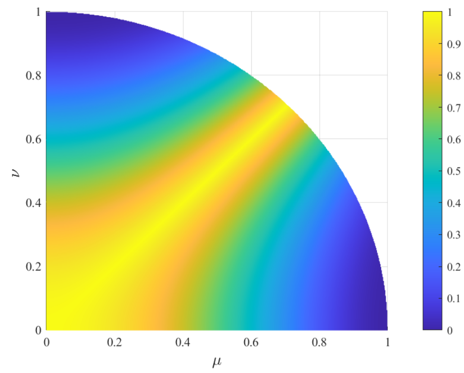
(a) $\mathcal{S}^1_{sin}$



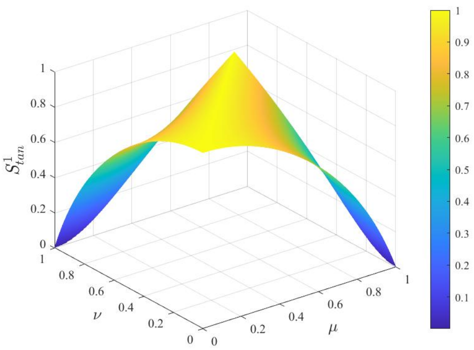
(b) $\mathcal{S}^1_{cos}$



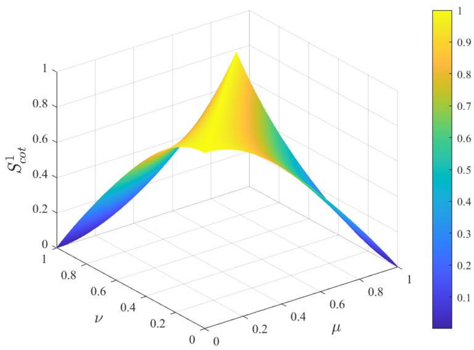
(c) Parameters of $\mathcal{S}^1_{sin}$



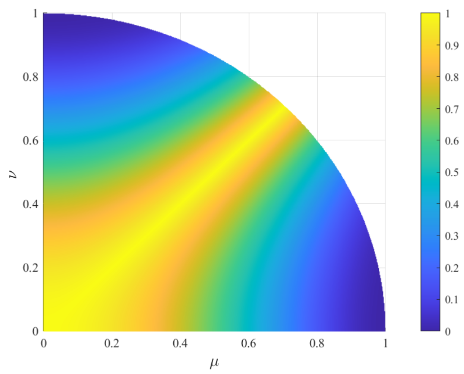
(d) Parameters of $\mathcal{S}^1_{cos}$



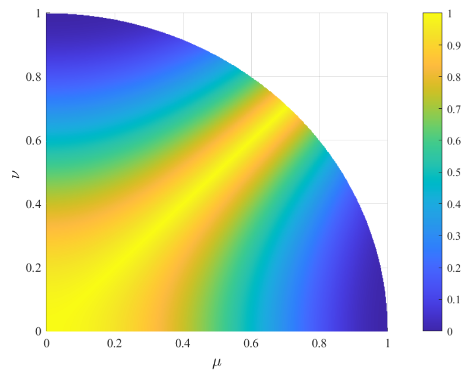
(e) $\mathcal{S}^1_{tan}$



(f) $\mathcal{S}^1_{cot}$



(g) Parameters of $\mathcal{S}^1_{tan}$



(h) Parameters of $\mathcal{S}^1_{cot}$

Figure 1: The similarity measures in Example 1

Similarly, we can obtain the results $S^1_{sin}(\mathbb{P}_2, \mathbb{P}_1) = 0.6392 = S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2)$, $S^1_{sin}(\mathbb{P}_3, \mathbb{P}_2) = 0.7437 = S^1_{sin}(\mathbb{P}_2, \mathbb{P}_3)$, $S^1_{sin}(\mathbb{P}_3, \mathbb{P}_1) = 0.4379 = S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3)$. *Also, it is obvious that* $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^1_{sin}(\mathbb{P}_1, \mathbb{P}_2)$ *and* $S^1_{sin}(\mathbb{P}_1, \mathbb{P}_3) \leq S^1_{sin}(\mathbb{P}_2, \mathbb{P}_3)$. *Thus, property 3 and property 4 of our introduced similarity measures are proved.*

5. Applications

In this section, we apply the trigonometric similarity measures to some decision-making problems.

5.1 *Description of decision-making problem*

Consider that $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$ is a finite UOD and there are m known patterns which are represented as CPFSs $\mathbb{Q}_j$ $(j = 1, 2, \ldots, m)$. The objective is to categorize the unknown patterns which are denoted as CPFSs $\mathbb{P}_t$ $(t = 1, 2, \ldots, s)$ based on its relationship with $\mathbb{Q}_j$ $(j = 1, 2, \ldots, m)$. The detailed process is outlined as follows:

Step 1 Calculate the similarity between $\mathbb{P}_t$ $(t = 1, 2, \ldots, s)$ and $\mathbb{Q}_j$ $(j = 1, 2, \ldots, m)$ through the introduced similarity measures or weighted similarity measures.

Step 2 Select the maximum similarity among the calculated results.

Step 3 Obtain the classification result of $\mathbb{P}_t$.

Algorithm 1 presents the corresponding official algorithmic process and the flowchart of decision-making process is shown in Fig. 2.

Algorithm 1 Algorithm for decision-making problems.

Input: A group of known patterns $\mathbb{Q}_j = \{\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_m\}$;
A group of unknown samples $\mathbb{P}_t = \{\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_s\}$;
Output: Classification of the unknown pattern $\mathbb{P}_t$

```
for t = 1; t ≤ s do
    /* Step 1 */
    for j = 1; j ≤ m do
        Compute the similarity S(P_t, Q_j) using Eq. 10- Eq. 25;
    end for
    /* Step 2 */
    Select the maximum similarity among the calculated results;
    /* Step 3 */
    Classify the unknown sample P_t;
end for
```

5.2 *Application in pattern recognition*

Example 3 *[9] There are four known patterns* $\mathbb{Q}_1$, $\mathbb{Q}_2$, $\mathbb{Q}_3$ *and* $\mathbb{Q}_4$ *in UOD* Φ, *which are represented by CPFSs as* $\mathbb{Q}_j = \{\langle \phi_i, \mathscr{X}_{\mathbb{P}_j}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathbb{Q}_j}(\phi_i)}}, \mathscr{Y}_{\mathbb{P}_j}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathbb{Q}_j}(\phi_i)}} \rangle | \phi_i \in \Phi\}$ $(j = 1, 2, 3, 4)$ *and the unknown pattern* $\mathbb{P} = \{\langle \phi_i, \mathscr{X}_{\mathbb{P}}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{X}_{\mathbb{P}}(\phi_i)}}, \mathscr{Y}_{\mathbb{P}}(\phi_i) \cdot e^{2\pi i \mathscr{W}_{\mathscr{Y}_{\mathbb{P}}(\phi_i)}} \rangle | \phi_i \in \Phi\}$. *The objective of the*

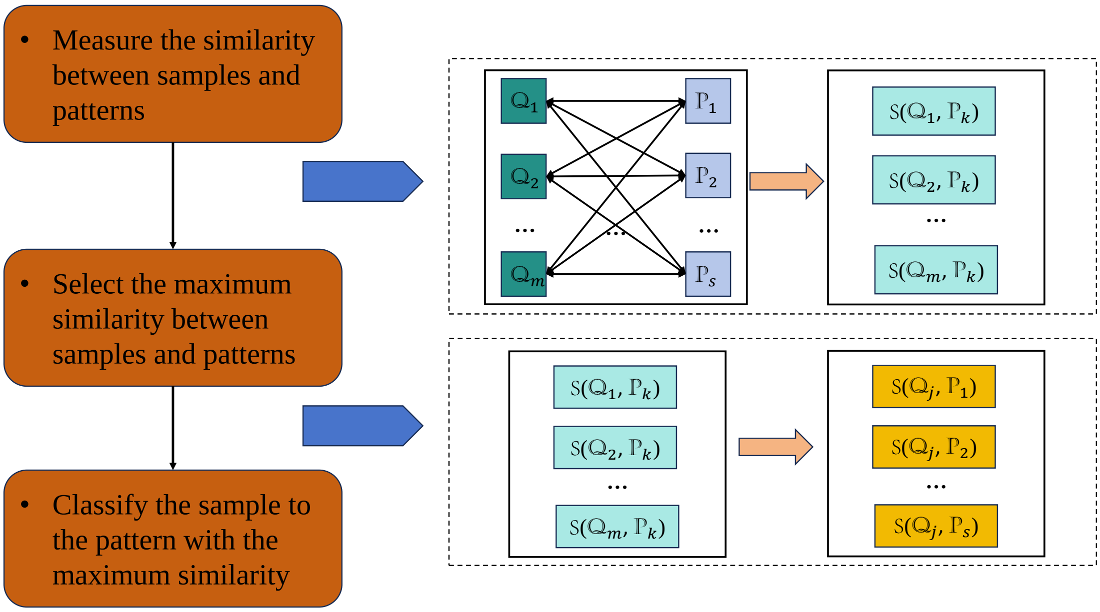


Figure 2: The flowchart of the decision-making process.

problem is to determine the class that $\mathbb{P}$ *belongs to.*

$$\mathbb{Q}_1 = \{(\phi_1, 0.2e^{2\pi i(0.3)}, 0.2e^{2\pi i(0.1)}), (\phi_2, 0.5e^{2\pi i(0.4)}, 0.0e^{2\pi i(0.1)}), (\phi_3, 0.1e^{2\pi i(0.3)}, 0.5e^{2\pi i(0.4)})\}$$
$$\mathbb{Q}_2 = \{(\phi_1, 0.4e^{2\pi i(0.2)}, 0.2e^{2\pi i(0.2)}), (\phi_2, 0.7e^{2\pi i(0.8)}, 0.0e^{2\pi i(0.1)}), (\phi_3, 0.1e^{2\pi i(0.1)}, 0.5e^{2\pi i(0.3)})\}$$
$$\mathbb{Q}_3 = \{(\phi_1, 0.4e^{2\pi i(0.5)}, 0.2e^{2\pi i(0.1)}), (\phi_2, 0.5e^{2\pi i(0.7)}, 0.0e^{2\pi i(0.2)}), (\phi_3, 0.1e^{2\pi i(0.2)}, 0.5e^{2\pi i(0.3)})\}$$
$$\mathbb{Q}_4 = \{(\phi_1, 0.6e^{2\pi i(0.4)}, 0.3e^{2\pi i(0.1)}), (\phi_2, 0.4e^{2\pi i(0.3)}, 0.0e^{2\pi i(0.1)}), (\phi_3, 0.1e^{2\pi i(0.3)}, 0.5e^{2\pi i(0.4)})\}$$
$$\mathbb{P} = \{(\phi_1, 0.3e^{2\pi i(0.5)}, 0.2e^{2\pi i(0.2)}), (\phi_2, 0.6e^{2\pi i(0.3)}, 0.0e^{2\pi i(0.2)}), (\phi_3, 0.2e^{2\pi i(0.1)}, 0.5e^{2\pi i(0.4)})\}$$

Considering the weight $\omega = \{0.3, 0.35, 0.35\}$, We employ various similarity measures to calculate the similarity between $\mathbb{P}$ and $\mathbb{Q}_j$. The computed results are depicted in Table 1, Table 2 and Fig. 3. According to the results, it is evident that $\mathbb{P}$ has the maximum similarity with $\mathbb{Q}_1$. All the introduced similarity measures and those used for comparison yield identical conclusions, indicating that the unknown pattern $\mathbb{P}$ belongs to $\mathbb{Q}_1$. Especially, we find that $S_{Wu}^{(1)}$ cannot compute the similarity, so it fails to classify the unknown pattern. The reason for this is that during the logarithmic calculation in $S_{Wu}^{(1)}$, a zero value is encountered. Logarithmic functions are undefined for zero and negative values in the real number domain, which led to the inability to compute the result. Therefore, its application in such a scenario is not feasible.

Furthermore, the degree of confidence(DoC) is employed to evaluate the effectiveness of various similarity measures which is defined as follows:

$$DoC = \sum_{j=1, j\neq j_0}^{m} |S(\mathbb{P}_j, \mathbb{P}) - S(\mathbb{P}_{j_0}, \mathbb{P})| \tag{26}$$

where $\mathbb{P}_{j_0}$ represents the classified result corresponding to $\mathbb{P}$. It is clear that a higher DoC indicates a better decision-making capability. In Fig. 4, we can see that the weighted similarity measures (WSimM) exhibit a higher DoC values compared to the unweighted similarity measure (SimM) which underscores the significance of prior knowledge in the decision-making problems.

Table 1: The results of similarity measures in Example 3.

Measures	$S(\mathbb{P}, \mathbb{Q}_1)$	$S(\mathbb{P}, \mathbb{Q}_2)$	$S(\mathbb{P}, \mathbb{Q}_3)$	$S(\mathbb{P}, \mathbb{Q}_4)$	Classification
S_{Wu}	**0.9117**	0.8233	0.8817	0.8700	$\mathbb{Q}_1$
S_{SK}	**0.9117**	0.8233	0.8817	0.8700	$\mathbb{Q}_1$
S_{WX}	**0.9533**	0.9092	0.9383	0.9350	$\mathbb{Q}_1$
S_G	**0.9167**	0.8233	0.8817	0.8833	$\mathbb{Q}_1$
S_{YC}	**0.9117**	0.8233	0.8817	0.8700	$\mathbb{Q}_1$
$S_{Wu}^{(1)}$	NaN	NaN	NaN	NaN	NaN
S_{sin}^1	**0.8618**	0.7315	0.8177	0.7984	$\mathbb{Q}_1$
S_{sin}^2	**0.8029**	0.6020	0.7309	0.6946	$\mathbb{Q}_1$
S_{cos}^1	**0.9895**	0.9445	0.9716	0.9736	$\mathbb{Q}_1$
S_{cos}^2	**0.9777**	0.8632	0.9323	0.9398	$\mathbb{Q}_1$
S_{tan}^1	**0.9305**	0.8589	0.9061	0.8972	$\mathbb{Q}_1$
S_{tan}^2	**0.9000**	0.7725	0.8550	0.8406	$\mathbb{Q}_1$
S_{cot}^1	**0.8707**	0.7653	0.8371	0.8180	$\mathbb{Q}_1$
S_{cot}^2	**0.8203**	0.6597	0.7670	0.7332	$\mathbb{Q}_1$

Note: Bold denotes the optimal solution.

Table 2: The results of weighted similarity measures in Example 3.

Measures	$S(\mathbb{P}, \mathbb{Q}_1)$	$S(\mathbb{P}, \mathbb{Q}_2)$	$S(\mathbb{P}, \mathbb{Q}_3)$	$S(\mathbb{P}, \mathbb{Q}_4)$	Classification
S_{wsin}^1	**0.8643**	0.7289	0.8125	0.8053	$\mathbb{Q}_1$
S_{wsin}^2	**0.8077**	0.5983	0.7229	0.7035	$\mathbb{Q}_1$
S_{wcos}^1	**0.9899**	0.9430	0.9703	0.9753	$\mathbb{Q}_1$
S_{wcos}^2	**0.9788**	0.8590	0.9292	0.9430	$\mathbb{Q}_1$
S_{wtan}^1	**0.9317**	0.8574	0.9034	0.9008	$\mathbb{Q}_1$
S_{wtan}^2	**0.9026**	0.7695	0.8505	0.8455	$\mathbb{Q}_1$
S_{wcot}^1	**0.8729**	0.7635	0.8327	0.8237	$\mathbb{Q}_1$
S_{wcot}^2	**0.8244**	0.6570	0.7605	0.7403	$\mathbb{Q}_1$

Note: Bold denotes the optimal solution.

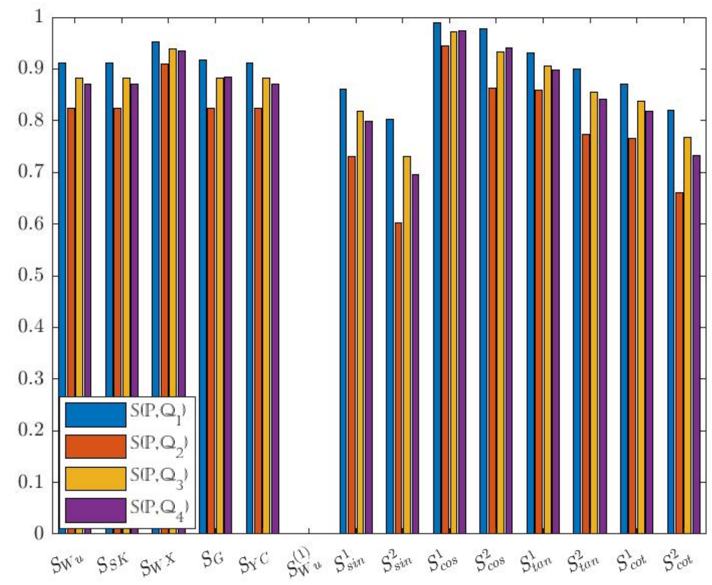


Figure 3: Results of different similarity measures in Example 3.

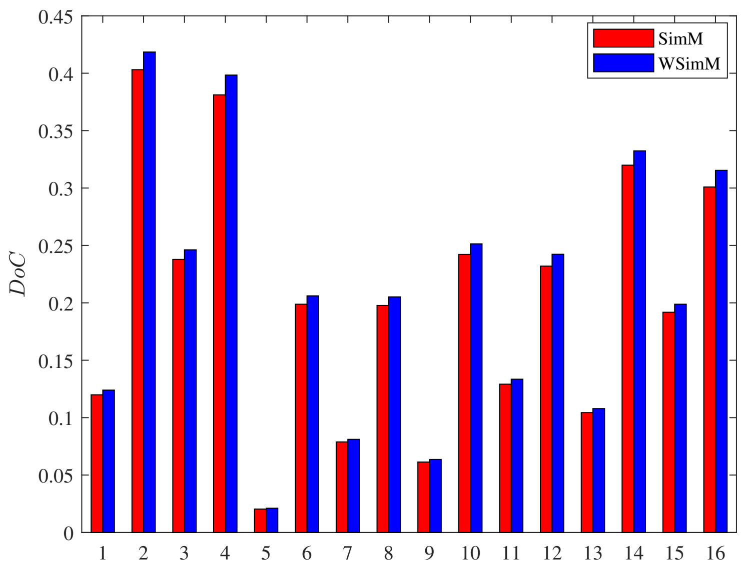


Figure 4: The DoC of different measures in Example 3.

5.3 Application in medical diagnosis

Example 4 *[36]Consider a medical diagnosis scenario for a patient $\mathbb{P}$ exhibiting symptoms $\Phi = \{$Temperature(ϕ_1), Headache(ϕ_2), Stomach pain(ϕ_3), Cough$(\phi_4)\}$. The characteristics of symptoms for potential medical disease which are represented as $\mathbb{P}_j = \{$Viral fever$(\mathbb{P}_1)$, Malaria$(\mathbb{P}_2)$, Typhoid$(\mathbb{P}_3)$, Stomach problem$(\mathbb{P}_4)\}$ are represented through the use of CPFSs. The complex Pythagorean fuzzy relationship from symptoms to disease and patient is depicted by CPFSs in Table 3.*

Table 3: CPFSs for disease and patient in Example 4.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\mathbb{P}_1$	$(\sqrt{0.8}e^{2\pi i\sqrt{0.7}}, \sqrt{0.1}e^{2\pi i\sqrt{0.2}})$	$(\sqrt{0.9}e^{2\pi i\sqrt{0.6}}, \sqrt{0.1}e^{2\pi i\sqrt{0.2}})$	$(\sqrt{0.7}e^{2\pi i\sqrt{0.8}}, \sqrt{0.2}e^{2\pi i\sqrt{0.1}})$	$(\sqrt{0.8}e^{2\pi i\sqrt{0.7}}, \sqrt{0.2}e^{2\pi i\sqrt{0.1}})$
$\mathbb{P}_2$	$(\sqrt{0.6}e^{2\pi i\sqrt{0.4}}, \sqrt{0.1}e^{2\pi i\sqrt{0.5}})$	$(\sqrt{0.4}e^{2\pi i\sqrt{0.9}}, \sqrt{0.5}e^{2\pi i\sqrt{0.1}})$	$(\sqrt{0.5}e^{2\pi i\sqrt{0.5}}, \sqrt{0.3}e^{2\pi i\sqrt{0.3}})$	$(\sqrt{0.4}e^{2\pi i\sqrt{0.9}}, \sqrt{0.5}e^{2\pi i\sqrt{0.1}})$
$\mathbb{P}_3$	$(\sqrt{0.3}e^{2\pi i\sqrt{0.8}}, \sqrt{0.3}e^{2\pi i\sqrt{0.1}})$	$(\sqrt{0.8}e^{2\pi i\sqrt{0.3}}, \sqrt{0.1}e^{2\pi i\sqrt{0.6}})$	$(\sqrt{0.7}e^{2\pi i\sqrt{0.6}}, \sqrt{0.2}e^{2\pi i\sqrt{0.2}})$	$(\sqrt{0.2}e^{2\pi i\sqrt{0.7}}, \sqrt{0.8}e^{2\pi i\sqrt{0.2}})$
$\mathbb{P}_4$	$(\sqrt{0.5}e^{2\pi i\sqrt{0.3}}, \sqrt{0.4}e^{2\pi i\sqrt{0.6}})$	$(\sqrt{0.3}e^{2\pi i\sqrt{0.1}}, \sqrt{0.6}e^{2\pi i\sqrt{0.3}})$	$(\sqrt{0.8}e^{2\pi i\sqrt{0.3}}, \sqrt{0.1}e^{2\pi i\sqrt{0.5}})$	$(\sqrt{0.1}e^{2\pi i\sqrt{0.3}}, \sqrt{0.6}e^{2\pi i\sqrt{0.5}})$
$\mathbb{P}$	$(\sqrt{0.8}e^{2\pi i\sqrt{0.6}}, \sqrt{0.1}e^{2\pi i\sqrt{0.2}})$	$(\sqrt{0.9}e^{2\pi i\sqrt{0.7}}, \sqrt{0.1}e^{2\pi i\sqrt{0.2}})$	$(\sqrt{0.7}e^{2\pi i\sqrt{0.8}}, \sqrt{0.2}e^{2\pi i\sqrt{0.1}})$	$(\sqrt{0.6}e^{2\pi i\sqrt{0.5}}, \sqrt{0.2}e^{2\pi i\sqrt{0.4}})$

Table 4: The results of similarity measures between $\mathbb{P}$ and $\mathbb{P}_t$ in Example 4.

Measures	$S(\mathbb{P}, \mathbb{P}_1)$	$S(\mathbb{P}, \mathbb{P}_2)$	$S(\mathbb{P}, \mathbb{P}_3)$	$S(\mathbb{P}, \mathbb{P}_4)$	Classification
S_{Wu}	**0.9125**	0.7000	0.7250	0.6000	$\mathbb{P}_1$
S_{SK}	**0.9125**	0.7000	0.7250	0.6000	$\mathbb{P}_1$
S_{WX}	**0.9438**	0.7563	0.7750	0.6625	$\mathbb{P}_1$
S_G	**0.9125**	0.7000	0.7250	0.6000	$\mathbb{P}_1$
S_{YC}	**0.9125**	0.7000	0.7250	0.6000	$\mathbb{P}_1$
$S_{Wu}^{(1)}$	**0.9254**	0.7190	0.7649	0.6633	$\mathbb{P}_1$
S_{sin}^1	**0.8651**	0.5474	0.5876	0.4230	$\mathbb{P}_1$
S_{sin}^2	**0.8083**	0.4493	0.3968	0.2972	$\mathbb{P}_1$
S_{cos}^1	**0.9794**	0.8883	0.8933	0.7960	$\mathbb{P}_1$
S_{cos}^2	**0.9666**	0.8245	0.7637	0.7028	$\mathbb{P}_1$
S_{tan}^1	**0.9306**	0.7595	0.7789	0.6716	$\mathbb{P}_1$
S_{tan}^2	**0.9006**	0.6952	0.6482	0.5843	$\mathbb{P}_1$
S_{cot}^1	**0.8792**	0.6141	0.6480	0.5144	$\mathbb{P}_1$
S_{cot}^2	**0.8302**	0.5373	0.4938	0.4157	$\mathbb{P}_1$

Note: Bold denotes the optimal solution.

We consider the weight vector $\omega = \{0.35, 0.25, 0.2, 0.2\}$, then the calculated results are shown in Table 4, Table 5 and Fig. 5. According to the results, we find that the value of $S(\mathbb{P}, \mathbb{P}_1)$ is the largest across all the introduced measures. This indicate that the patient $\mathbb{P}$ is diagnosis with Viral fever ($\mathbb{P}_1$). The DoC values of different measures are depicted in Fig. 6. We can observe that the weighted similarity measures (WSimM) demonstrate better performance in comparison to the unweighted similarity measures (SimM), highlighting the significance of incorporating relevant prior knowledge into decision-making scenarios. What is more, the similarity measures S_{wsin}^2 and S_{wsin}^4 especially exhibit a high DoC value, which implies that these similarity measures are the most dependable in the medical diagnosis problems. This will assist decision-makers in identifying the most trustworthy decisions within this specific context.

Table 5: The results of weighted similarity measures between $\mathbb{P}$ and $\mathbb{P}_t$ in Example 4.

Measures	$S(\mathbb{P},\mathbb{P}_1)$	$S(\mathbb{P},\mathbb{P}_2)$	$S(\mathbb{P},\mathbb{P}_3)$	$S(\mathbb{P},\mathbb{P}_4)$	Classification
S^1_{wsin}	**0.8764**	0.5544	0.5726	0.4196	$\mathbb{P}_1$
S^2_{wsin}	**0.8153**	0.4560	0.3820	0.3092	$\mathbb{P}_1$
S^1_{wcos}	**0.9829**	0.8918	0.8887	0.7941	$\mathbb{P}_1$
S^2_{wcos}	**0.9708**	0.8286	0.7575	0.7129	$\mathbb{P}_1$
S^1_{wtan}	**0.9366**	0.7637	0.7708	0.6695	$\mathbb{P}_1$
S^2_{wtan}	**0.9048**	0.6994	0.6402	0.5932	$\mathbb{P}_1$
S^1_{wcot}	**0.8883**	0.6195	0.6358	0.5118	$\mathbb{P}_1$
S^2_{wcot}	**0.8350**	0.5425	0.4826	0.4252	$\mathbb{P}_1$

Note: Bold denotes the optimal solution.

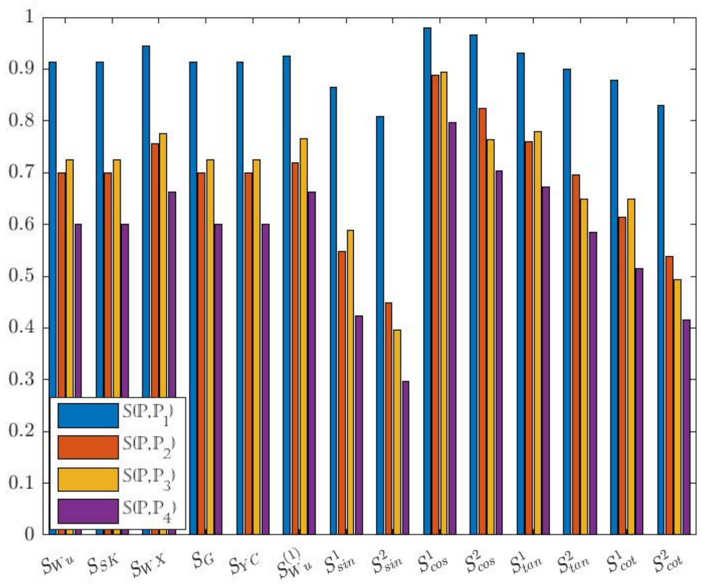


Figure 5: Results of different measures in Example 4.

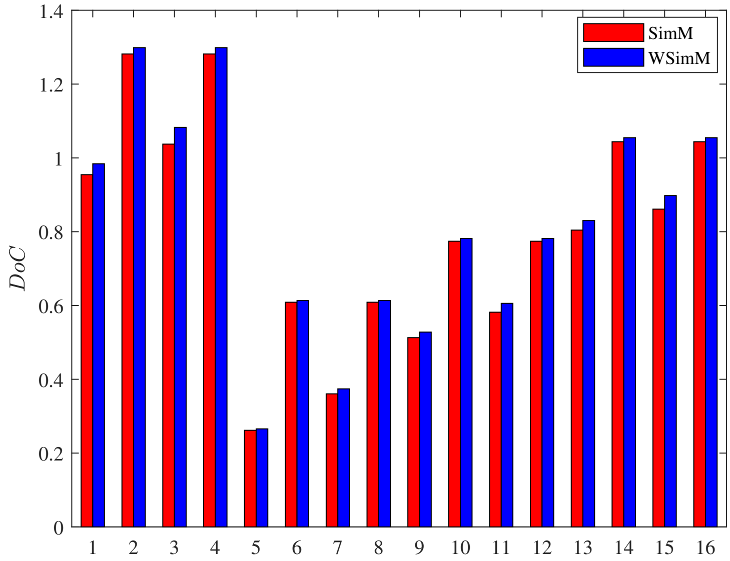


Figure 6: The DoC of different measures in Example 4.

6. Conclusions

In this paper, we present a set of similarity measures and weighted similarity measures based on trigonometric functions, including sine function, cosine function, tangent function and cotangent function. Then, We confirm that the proposed measures fulfill essential properties and exhibit its accuracy and efficacy through extensive numerical experiments. Furthermore, we apply these trigonometric similarity measures to pattern recognition and medical diagnosis. The results demonstrate that our proposed similarity measures are capable of achieving favorable outcomes in these applications. In the future, we can present the method of adaptive weights based on the proposed similarity measures to cope with different decision-making contexts. Additionally, trigonometric similarity measures are expected to merit consideration for application in interval-valued Pythagorean fuzzy sets and complex interval-valued Pythagorean fuzzy sets which will boost the precision of outcomes by express data in interval format, particularly when handling uncertain data, and carries significant implications for both research endeavors and practical implementations.

Acknowledgement

This research was not funded by any grant.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Akram, M., Zahid, K., & Kahraman, C. (2023). New optimization technique for group decision analysis with complex pythagorean fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, *44*(3), 3621–3645. https://doi.org/10.3233/JIFS-220764

[2] Ali, M. Y. (2023). Some trigonometric similarity measures of complex fuzzy sets with application. *Ural Mathematical Journal*, *9*(1), 18–28. https://doi.org/10.15826/umj.2023.1.002

[3] Ali, Z., Emam, W., Mahmood, T., & Wang, H. (2024). Archimedean heronian mean operators based on complex intuitionistic fuzzy sets and their applications in decision-making problems. *Heliyon*, *10*(3). https://doi.org/10.1016/j.heliyon.2024.e24767
[4] Alkouri, A. (J. S., & Salleh, A. R. (2012). Complex intuitionistic fuzzy sets. *AIP Conference Proceedings*, *1482*(1), 464–470. https://doi.org/10.1063/1.4757515
[5] Arora, H. D., Kumar, V., & Naithani, A. (2024). Impact of trigonometric similarity measures for pythagorean fuzzy sets and their applications. *Yugoslav Journal of Operations Research*. https://doi.org/10.2298/YJOR220515004A
[6] Asif, M., Ishtiaq, U., & Argyros, I. K. (2025). Hamacher aggregation operators for pythagorean fuzzy set and its application in multi-attribute decision-making problem. *Spectrum of Operational Research*, *2*(1), 27–40. https://doi.org/0000-0002-5228-1073
[7] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87–96. https://doi.org/10.1007/978-3-7908-1870-3_1
[8] Garg, H., Kahraman, C., Ali, Z., & Mahmood, T. (2023). Interaction hamy mean operators for complex pythagorean fuzzy information and their applications to security threats in computers. *Journal of Intelligent & Fuzzy Systems*, *44*(3), 4459–4479. https://doi.org/10.3233/JIFS-220947
[9] Garg, H., & Rani, D. (2019). Some results on information measures for complex intuitionistic fuzzy sets. *International Journal of Intelligent Systems*, *34*(10), 2319–2363. https://doi.org/10.1002/int.22127
[10] Garg, H., & Rani, D. (2020). Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process. *Artificial Intelligence Review*, *53*, 3595–3620. https://doi.org/10.1007/s10462-019-09772-x
[11] Gogoi, S., Gohain, B., & Chutia, R. (2023). Distance measures on intuitionistic fuzzy sets based on cross-information dissimilarity and their diverse applications. *Artificial Intelligence Review*, *56*(Suppl 3), 3471–3514. https://doi.org/10.1007/s10462-023-10608-y
[12] Hezam, I. M., Rahman, K., Alshamrani, A., & Božanić, D. (2023). Geometric aggregation operators for solving multicriteria group decision-making problems based on complex pythagorean fuzzy sets. *Symmetry*, *15*(4), 826. https://doi.org/10.3390/sym15040826
[13] Hussain, Z., Alam, S., Hussain, R., & ur Rahman, S. (2024). New similarity measure of pythagorean fuzzy sets based on the jaccard index with its application to clustering. *Ain Shams Engineering Journal*, *15*(1), 102294. https://doi.org/10.1016/j.asej.2023.102294
[14] Krishankumar, R., Ravichandran, K., Aggarwal, M., & Pamucar, D. (2023). An improved entropy function for the intuitionistic fuzzy sets with application to cloud vendor selection. *Decision Analytics Journal*, *7*, 100262. https://doi.org/10.1016/j.dajour.2023.100262
[15] Labassi, F., ur Rehman, U., Alsuraiheed, T., Mahmood, T., & Khan, M. A. (2024). A novel approach towards complex pythagorean fuzzy sets and their applications in visualization technology. *IEEE Access*. https://doi.org/10.1109/ACCESS.2024.3393138
[16] Liao, Y., & Peng, X. (2024). Pythagorean fuzzy information measure with applications in multicriteria decision-making and medical diagnosis. *Engineering Applications of Artificial Intelligence*, *136*, 108927. https://doi.org/10.1016/j.engappai.2024.108927
[17] Liu, P., Ali, Z., & Ding, J. (2024). Power dombi aggregation operators for complex pythagorean fuzzy sets and their applications in green supply chain management. *International Journal of Fuzzy Systems*, 1–16. https://doi.org/10.1007/s40815-024-01691-6
[18] Liu, P., Ali, Z., & Mahmood, T. (2023). Archimedean aggregation operators based on complex pythagorean fuzzy sets using confidence levels and their application in decision making. *International Journal of Fuzzy Systems*, *25*(1), 42–58. https://doi.org/10.1016/j.engappai.2023.107153

[19] Liu, P., Ali, Z., Mahmood, T., & Geng, Y. (2023). Prioritized aggregation operators for complex intuitionistic fuzzy sets based on aczel-alsina t-norm and t-conorm and their applications in decision-making. *International Journal of Fuzzy Systems*, *25*(7), 2590–2608. https://doi.org/10.1007/s40815-023-01541-x

[20] Liu, Z. (2023). An effective conflict management method based on belief similarity measure and entropy for multi-sensor data fusion. *Artificial Intelligence Review*, *56*(12), 15495–15522. https://doi.org/10.1007/s10462-023-10533-0

[21] Liu, Z. (2024). Hellinger distance measures on pythagorean fuzzy environment via their applications. *International Journal of Knowledge-based and Intelligent Engineering Systems*, *28*(2), 211–229. https://doi.org/10.3233/KES-230150

[22] Liu, Z., Deveci, M., Pamučar, D., & Pedrycz, W. (2024). An effective multi-source data fusion approach based on α-divergence in belief functions theory with applications to air target recognition and fault diagnosis. *Information Fusion*, *110*, 102458. https://doi.org/10.1016/j.inffus.2024.102458

[23] Liu, Z., Hezam, I. M., Letchmunan, S., Qiu, H., & Alshamrani, A. M. (2024). Generalized similarity measure for multisensor information fusion via dempster-shafer evidence theory. *IEEE Access*, *12*, 104629–104642. https://doi.org/10.1109/ACCESS.2024.3435459

[24] Liu, Z., & Letchmunan, S. (2024). Enhanced fuzzy clustering for incomplete instance with evidence combination. *ACM Transactions on Knowledge Discovery from Data*, *18*(3), 1–20. https://doi.org/10.1145/3638061

[25] Liu, Z., Qiu, H., Deveci, M., Pedrycz, W., & Siarry, P. (2024). Multi-view neutrosophic c-means clustering algorithms. *Expert Systems with Applications*, 125454. https://doi.org/10.1016/j.eswa.2024.125454

[26] Liu, Z., Qiu, H., & Letchmunan, S. (2024). Self-adaptive attribute weighted neutrosophic c-means clustering for biomedical applications. *Alexandria Engineering Journal*, *96*, 42–57. https://doi.org/10.1016/j.aej.2024.03.092

[27] Liu, Z., Wang, D., Letchmunan, S., Aljohani, S., & Mlaiki, N. (2024). Elementary function-based fermatean fuzzy similarity measures: Applications to medical pattern recognition and multi-criteria decision-making. *IEEE Access*, *12*, 163452–163464. https://doi.org/10.1109/ACCESS.2024.3490606

[28] Mahmood, T., & Ali, Z. (2023). Multi-attribute decision-making methods based on aczel–alsina power aggregation operators for managing complex intuitionistic fuzzy sets. *Computational and Applied Mathematics*, *42*(2), 87. https://doi.org/10.1007/s40314-023-02204-1

[29] Ngan, S.-C. (2024). An extension framework for creating operators and functions for intuitionistic fuzzy sets. *Information Sciences*, *666*, 120336. https://doi.org/10.1016/j.ins.2024.120336

[30] Qiu, H., Liu, Z., & Letchmunan, S. (2024). Incm: Neutrosophic c-means clustering algorithm for interval-valued data. *Granular Computing*, *9*(2), 34. https://doi.org/10.1007/s41066-024-00452-y

[31] Rahman, K., Garg, H., Ali, R., Alfalqi, S. H., & Lamoudan, T. (2023). Algorithms for decision-making process using complex pythagorean fuzzy set and its application to hospital siting for covid-19 patients. *Engineering Applications of Artificial Intelligence*, *126*, 107153. https://doi.org/10.1016/j.engappai.2023.107153

[32] Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. *IEEE transactions on fuzzy systems*, *10*(2), 171–186. https://doi.org/10.1109/91.995119

[33] Rani, D., & Garg, H. (2017). Distance measures between the complex intuitionistic fuzzy sets and their applications to the decision-making process. *International Journal for Uncertainty Quantification*, *7*(5). https://doi.org/10.1615/Int.J.UncertaintyQuantification.2017020356

[34] Ullah, K., Mahmood, T., Ali, Z., & Jan, N. (2020). On some distance measures of complex pythagorean fuzzy sets and their applications in pattern recognition. *Complex & Intelligent Systems*, *6*, 15–27. https://doi.org/10.1007/s40747-019-0103-6
[35] Wang, D., Yuan, Y., Liu, Z., Zhu, S., & Sun, Z. (2024). Novel distance measures of q-rung orthopair fuzzy sets and their applications. *Symmetry*, *16*(5), 574. https://doi.org/10.3390/sym16050574
[36] Wu, D., Zhu, Z., Ullah, K., Liu, L., Wu, X., & Zhang, X. (2023). Analysis of hamming and hausdorff 3d distance measures for complex pythagorean fuzzy sets and their applications in pattern recognition and medical diagnosis. *Complex & Intelligent Systems*, *9*(4), 4147–4158. https://doi.org/10.1007/s40747-022-00939-8
[37] Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, *22*(4), 958–965. https://doi.org/10.1109/TFUZZ.2013.2278989
[38] Yang, Y., Zhang, D., Wang, Y., & Zhang, Y. (2018). Similarity measure of intuitionistic fuzzy sets based on sine function and its application. *Computer Engineering & Science*, *40*(01), 133. https://doi.org/http://joces.nudt.edu.cn/EN/Y2018/V40/I01/133
[39] Ye, J. (2016). Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes. *Journal of Intelligent & Fuzzy Systems*, *30*(1), 151–158. https://doi.org/0.3233/IFS-151741
[40] Zadeh, L. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
[41] Zhou, Y., Ejegwa, P. A., & Johnny, S. E. (2023). Generalized similarity operator for intuitionistic fuzzy sets and its applications based on recognition principle and multiple criteria decision making technique. *International Journal of Computational Intelligence Systems*, *16*(1), 85. https://doi.org/10.1007/s44196-023-00245-2